Amplituhedra Across m: A Map of the Known and Unknown

Big Picture for ATOP

(P) proved/established **(C)** conjectured **(O)** open/unknown

Direction	m = 1	m = 2	m = 4	m > 4
1. Image	**(P)** Bounded complex of a cyclic hyperplane arrangement; facets $\langle Y i\rangle = 0$; the sequence $\langle Y1\rangle, \langle Y2\rangle, \ldots, \langle Yn\rangle$ has k sign flips [1].	**(P)** Facets $\langle Y\, i\, i{+}1\rangle = 0$; the full face stratification is known [2]. The sequence $\langle Y12\rangle, \langle Y13\rangle, \ldots, \langle Y1n\rangle$ has k sign flips & $\langle Y i, i+1\rangle > 0$ [3].	**(P)** Facets $\langle Y\, i\, i{+}1\, j\, j{+}1\rangle = 0$. The sequence $\langle Y1234\rangle, \langle Y1235\rangle, \ldots, \langle Y123n\rangle$ has k sign flips & $\langle Y i, i+1, j, j+1\rangle > 0$ [18,24]. **(O)** Full face stratification is open.	**(P)** For even m, facets: $\langle Y\, i_1\, i_1{+}1 \cdots i_{m/2}\, i_{m/2}{+}1\rangle = 0$ & more facets of higher-degree in the twistor coordinates [17]. **(O)** Full list of facets. **(O)** The naive general sign-flip description fails; a replacement and the full face stratification are open.
2. Fibers / tiles	**(P)** Tiles are classified and lie among the $\mathrm{IN}_1 = 1$ cells; there are $\mathrm{IN}_1 = 1$ cells which are not tiles. [1,16]	**(P)** Tiles are exactly the $\mathrm{IN}_2 = 1$ cells and are encoded by bicolored subdivisions of the n-gon [3].	**(P)** BCFW tiles & few other examples, lie among the $\mathrm{IN}_4 = 1$ cells [6,7,21]. **(O)** General classifications of tiles and $\mathrm{IN}_m = 1$ cells are open.	**(P)** Classification of $\mathrm{IN}_m = 1$ cells with tree plabic graphs; fibers of amplituhedron map via vector-relation configurations (m-VRCs); plabic tangles to generate $\mathrm{IN}_m = 1$ cells recursively [9]. **(O)** General classifications of tiles and $\mathrm{IN}_m = 1$ cells are open.
3. Tilings	**(P)** Distinguished positroid images give the bounded-complex cell decomposition [1,16].	**(P)** Tilings in bijection with positroid tilings of $\Delta_{k+1,n}$ [3]. All tilings have $\binom{n-2}{k}$-many tiles (Magic Number theorem). **(O)** Characterize combinatorial conditions for a collection of tiles to give a tiling [4].	**(P)** Every iterated BCFW recursion gives a BCFW tiling; non-BCFW tilings also occur [5–7,21]. **(O)** Finding new recursions beyond BCFW. **(C)** (Magic number conjecture): all tilings have Narayana-many tiles [16].	**(P)** $k = 1$ tilings are cyclic-polytope triangulations. **(C)** Conjectural BCFW tiling for $(k, m) = (2, 6)$. **(C)** When $m > 4$, for sufficiently large n and k, there are no tilings [17].
4. Dualities	N/A	**(P)** T-duality $\mathcal{A}_{n,k,2} \leftrightarrow \Delta_{k+1,n}$ gives bijections of tiles and tilings. [3]	**(C)** T-duality with the momentum amplituhedron: bijection between tiles and tilings [19]. **(P)** Proved for BCFW tiles and tilings [18].	**(P)** Parity duality for even m: bijection between tiles and tilings for $\mathcal{A}_{n,k,m}$ and $\mathcal{A}_{n,n-m-k,m}$ [8]. **(O)** T-duality for $m > 4$ remains open [19].

Each dotted line is deliberately left for corrections, new statements, or references arising during the workshop.
All indices are understood cyclically, with the twisted-cyclic convention $Z_{i+n} := (-1)^{k+m-1} Z_i$. Sign regions and all semi-algebraic descriptions are understood with their closures.

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Continuation

(P) proved/established **(C)** conjectured **(O)** open/unknown

Direction	m = 1	m = 2	m = 4	m > 4
5. Cluster algebra	Trivial.	**(P)** Tiles are signed regions described by compatible cluster variables of $\mathrm{Gr}(2,n)$ (type A_{n-3}). [3]	**(P)** BCFW tiles are signed regions described by compatible cluster variables of $\mathrm{Gr}(4,n)$. [6,7,21]	**(C)** Tiles are signed regions described by compatible cluster variables of $\mathrm{Gr}(m,n)$. **(C)** 'nice' plabic tangles built from $\mathrm{IN}_m = 1$ cells produce *cluster promotion maps*; **(P)** proved for several families. [9]
6. Positive geometry	**(P)** The cyclic hyperplane arrangement is a positive geometry [1,10].	**(P)** Proved for $k = 2$ [12]. **(C)** Conjectural canonical form for every tile [20].	**(C)** Conjectural canonical forms for BCFW tiles [6,7,11].	**(O)** No general positive-geometry/canonical-form theorem is known.
7. Physics	N/A	Appears in Wilson-loop/form-factor constructions [13]. For $k = 2$, it describes the one-loop MHV geometry.	Tree amplitudes in planar $\mathcal{N} = 4$ SYM [14]; tree amplituhedra also crucial for leading singularities and Landau loci [15,22].	**(O)** Generalized scattering amplitudes? Possible connection with generalized biadjoint scalar amplitudes? [23]
Additions from ATOP				

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