

arXiv: 2408.1156

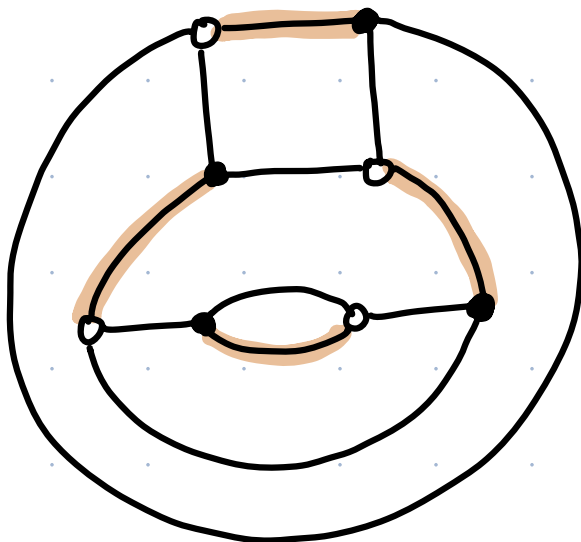
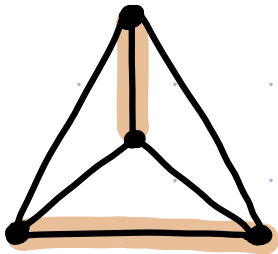
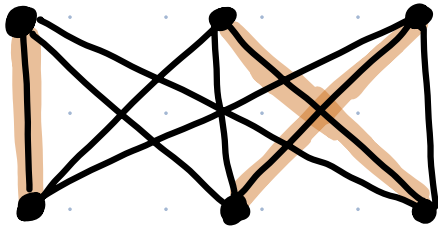
Dimer face polynomials

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Slides at bit.ly/msherben under "Talks"

Dimers



Defn: A dimer or perfect matching of G is a subset of edges covering ea. vertex exactly once.

• $\mathcal{D}_G$ = set of dimers

Q: What is $|\mathcal{D}_G|$?

• Computationally difficult in general

• If G is planar, properly bicolored,
[Kasteleyn]

drawn in plane
w/ no edge crossing

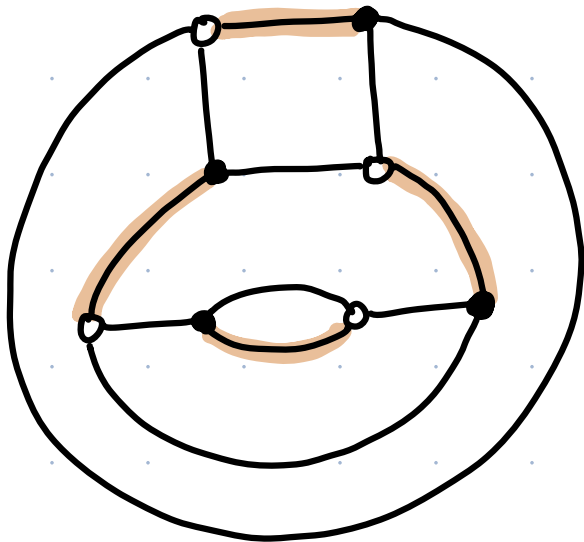
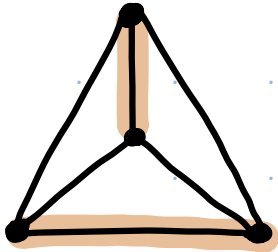
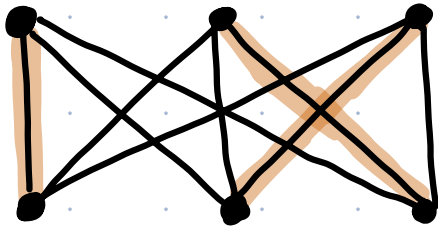
no , only 

Then $|\mathcal{D}_G| = \det A$.

white vt  black vt 

$A_{wb} = \begin{cases} \pm 1 & (w,b) \text{ an edge} \\ 0 & \text{else} \end{cases}$

Dimers



Defn: A dimer or perfect matching of G is a subset of edges covering ea. vertex exactly once.

• $\mathcal{D}_G$ = set of dimers

Q: What is $|\mathcal{D}_G|$?

- Computationally difficult in general
- If G is planar, properly bicolored, [Kasteleyn] then $|\mathcal{D}_G| = \det A$.

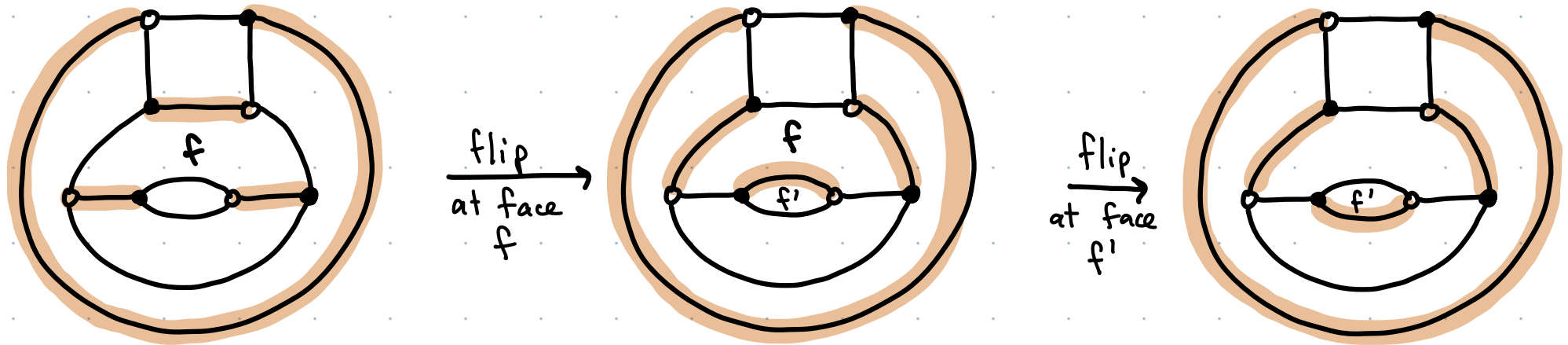
Q': Generating polynomial for $\mathcal{D}_G$?

$$\sum_{M \in \mathcal{D}_G} \text{wt}(M)$$

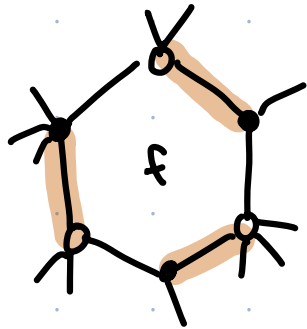
↖ if $\text{wt}(M) = \prod_{e \in M} z_e$, get partition function

More structure on $\mathcal{D}_G$

- G planar, connected, properly bicolored
- Local move on dimers: "flips" at (finite) faces

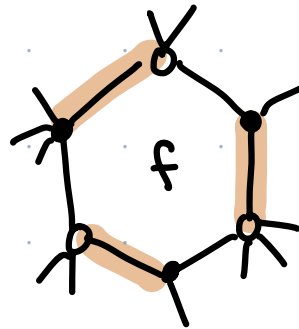


Defn:



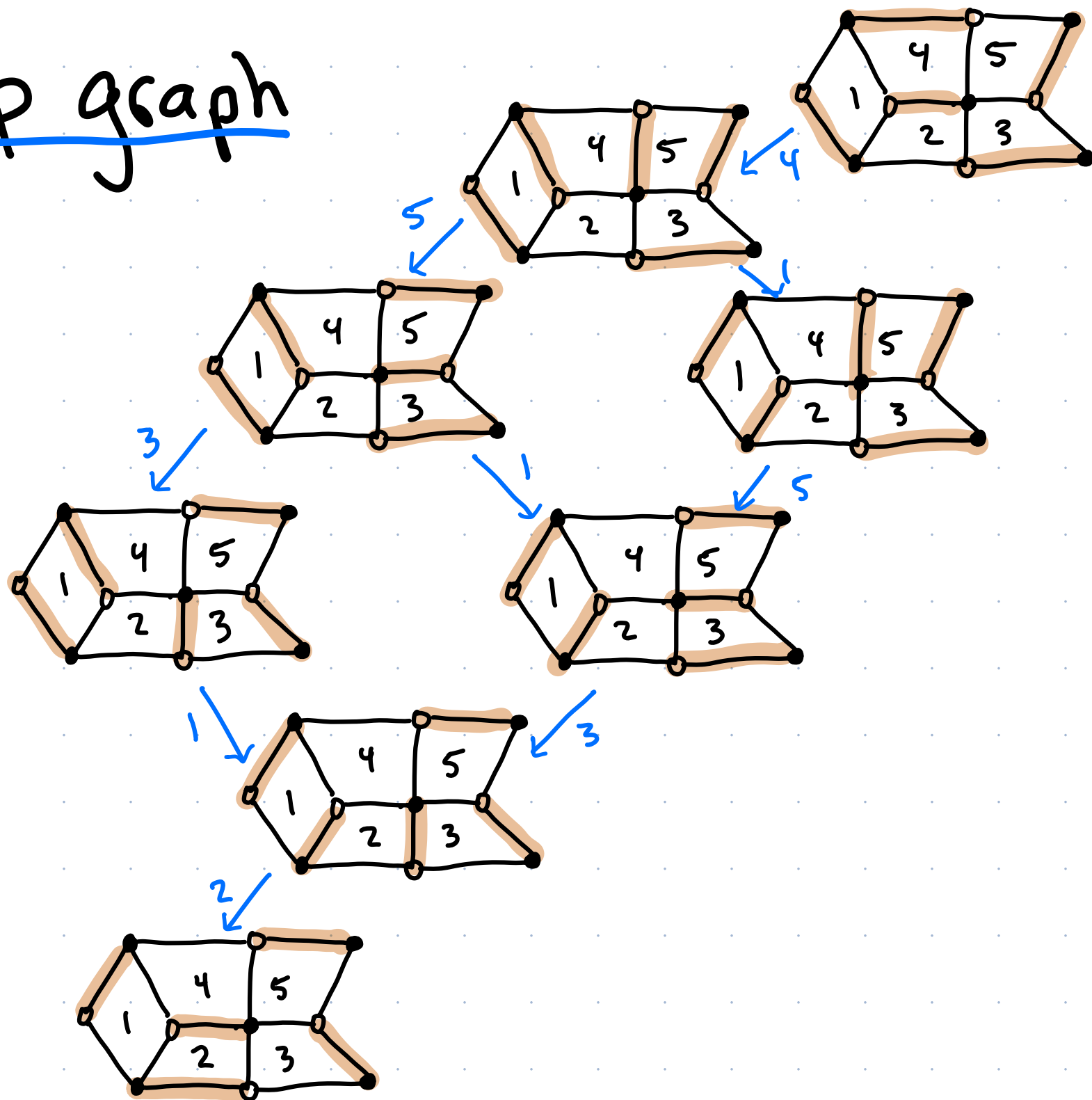
orienting f clockwise,
edges in M are $\bullet \rightarrow \bullet$

up-flip
down-flip



orienting f clockwise,
edges in M' are $\bullet \rightarrow \bullet$

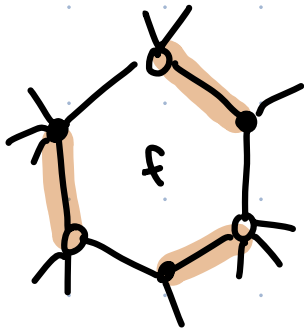
Flip graph



Dimer lattice

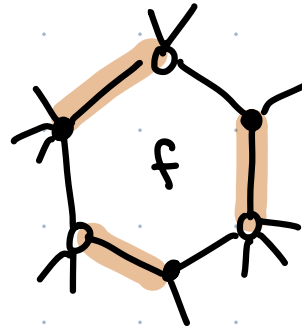
- G connected plane graph, properly bicolored, every edge in matching $\}^{(*)}$
 $\leadsto G$ a $(*)$ -graph.

Defn:



orienting f clockwise,
edges in M are $\bullet \rightarrow \circ$

up-flip
 $\longleftrightarrow$
 down-flip

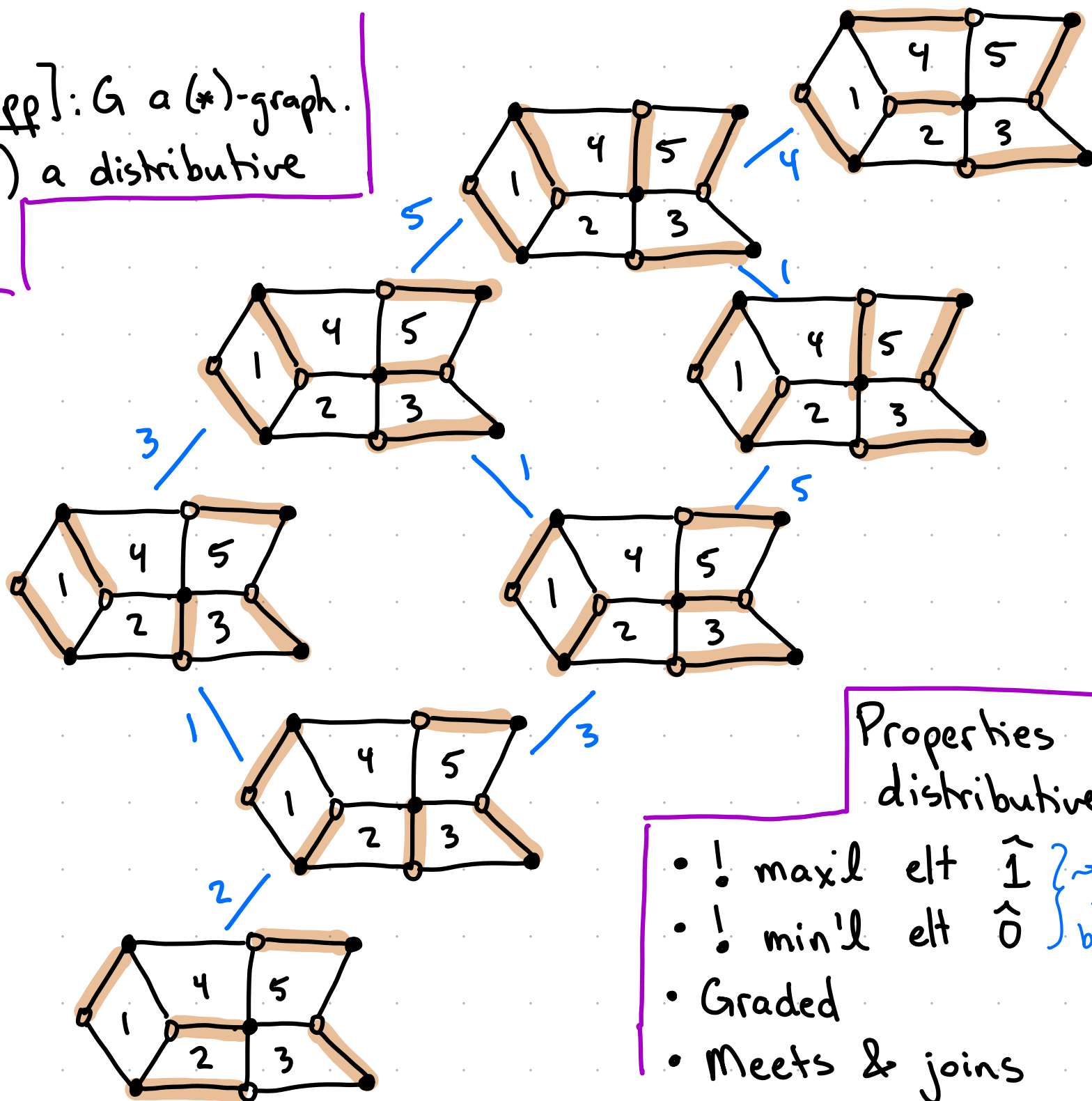


orienting f clockwise,
edges in M' are $\circ \rightarrow \bullet$

Thm [Propp '02]: G a $(*)$ -graph. Define a relation on $\mathcal{D}_G$ by:
 $M \leq M'$ if $\exists$ seq. of up-flips from M to M' .

Then $(\mathcal{D}_G, \leq)$ is a distributive lattice.

Thm [Prop]: G a $(*)$ -graph.
 $(\mathcal{D}_G, \leq)$ a distributive
 lattice

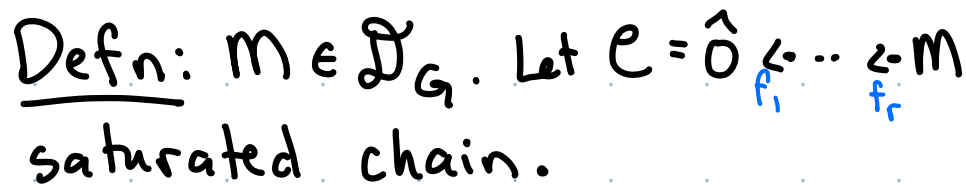


Properties of
 distributive lattices:

- ! max'l elt $\hat{1}$
 - ! min'l elt $\hat{0}$
 - Graded
 - Meets & joins
- alg. for flip seq. b/w any m, m'*

← Generating fcn that knows about dimer lattice

- Lem: $M \in \mathcal{D}_G$. All saturated chains $\hat{O} \leftarrow m_1 \leftarrow m_2 \leftarrow \dots \leftarrow M$ have the same multi-set of labels.



$$\text{mrk}(m) = y_{f_1} y_{f_2} \cdots y_{f_r}$$

The dimer face polynomial is

$$D_G := \sum_{m \in \mathcal{D}_G} \text{mrk}(m)$$

e.g. $D_G = 1 + y_2 + y_1 y_2 + y_2 y_3 + y_1 y_2 y_3 + y_2 y_3 y_5 + \dots$

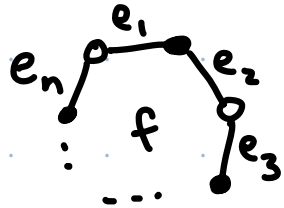
Rel'n to other dimer generating fns

- Partition fcn
(aka edge-generating fcn)

$$Z_G := \sum_{m \in \mathcal{D}_G} \left(\prod_{e \in m} z_e \right)$$

↳ Well-studied in statistical mechanics, probability, graph thry, etc.

- Invertible change of variables from $\mathcal{D}_G$ to Z_G :



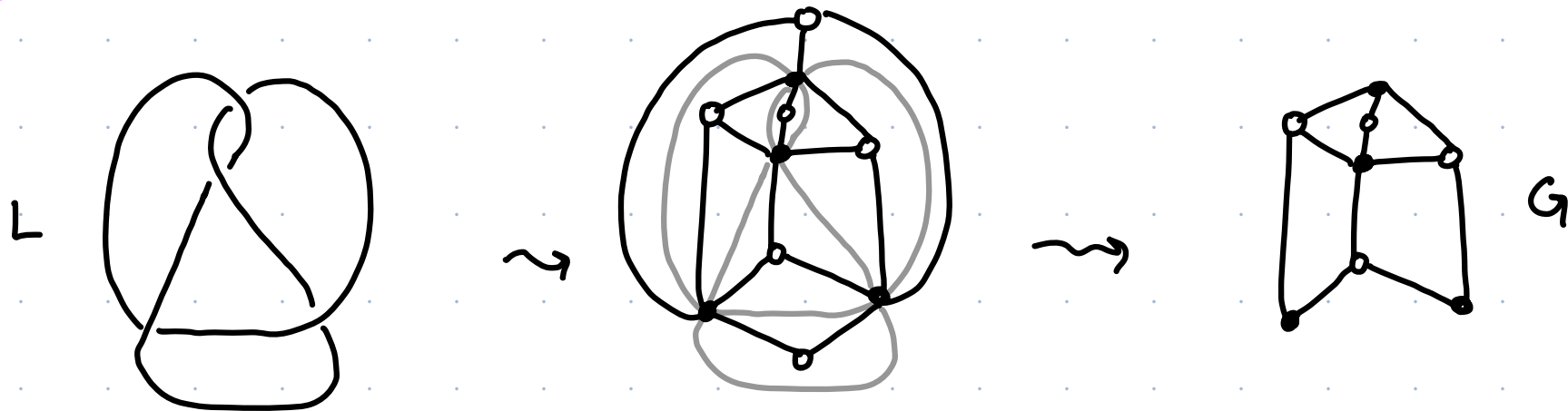
$$y_f \mapsto \frac{z_{e_2} z_{e_4} \dots z_{e_n}}{z_{e_1} z_{e_3} \dots z_{e_{n-1}}}$$

$$\text{sends } \mathcal{D}_G \mapsto \frac{1}{\prod_{e \in \hat{\mathcal{D}}} z_e} Z_G.$$

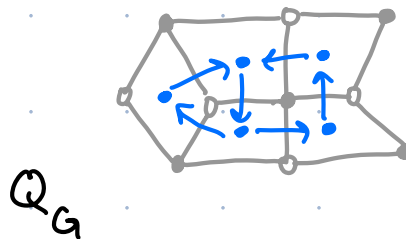
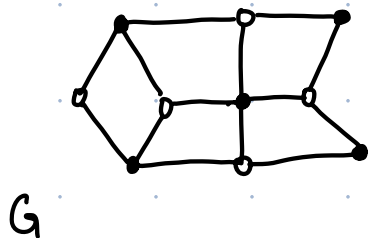
- $\mathcal{D}_G$ uses $|V|-2$ fewer variables, records poset structure.

Results on dimer face polynomials

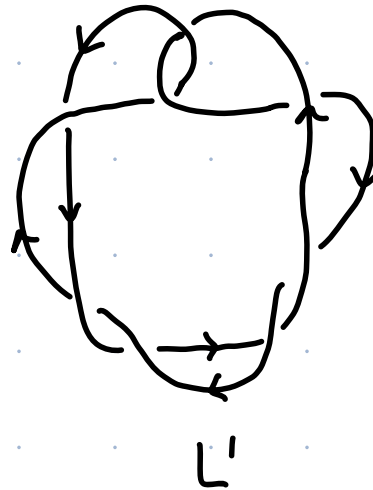
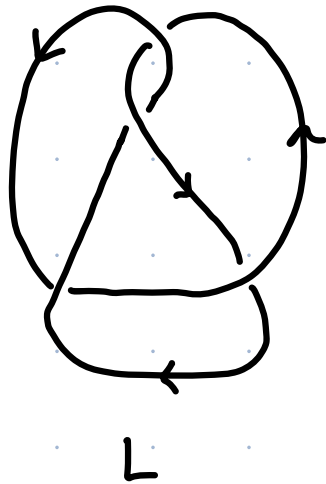
Thm: [mmsBV] If G comes from a link diagram L , then D_G specializes to Alexander polynomial of L .



Thm: [mmsBV] D_G is an F-polynomial in the cluster algebra $A(Q_G)$. $\rightsquigarrow$ Gives (many) recursive formulas for D_G in terms of $D_{G'}$



Links & diagrams



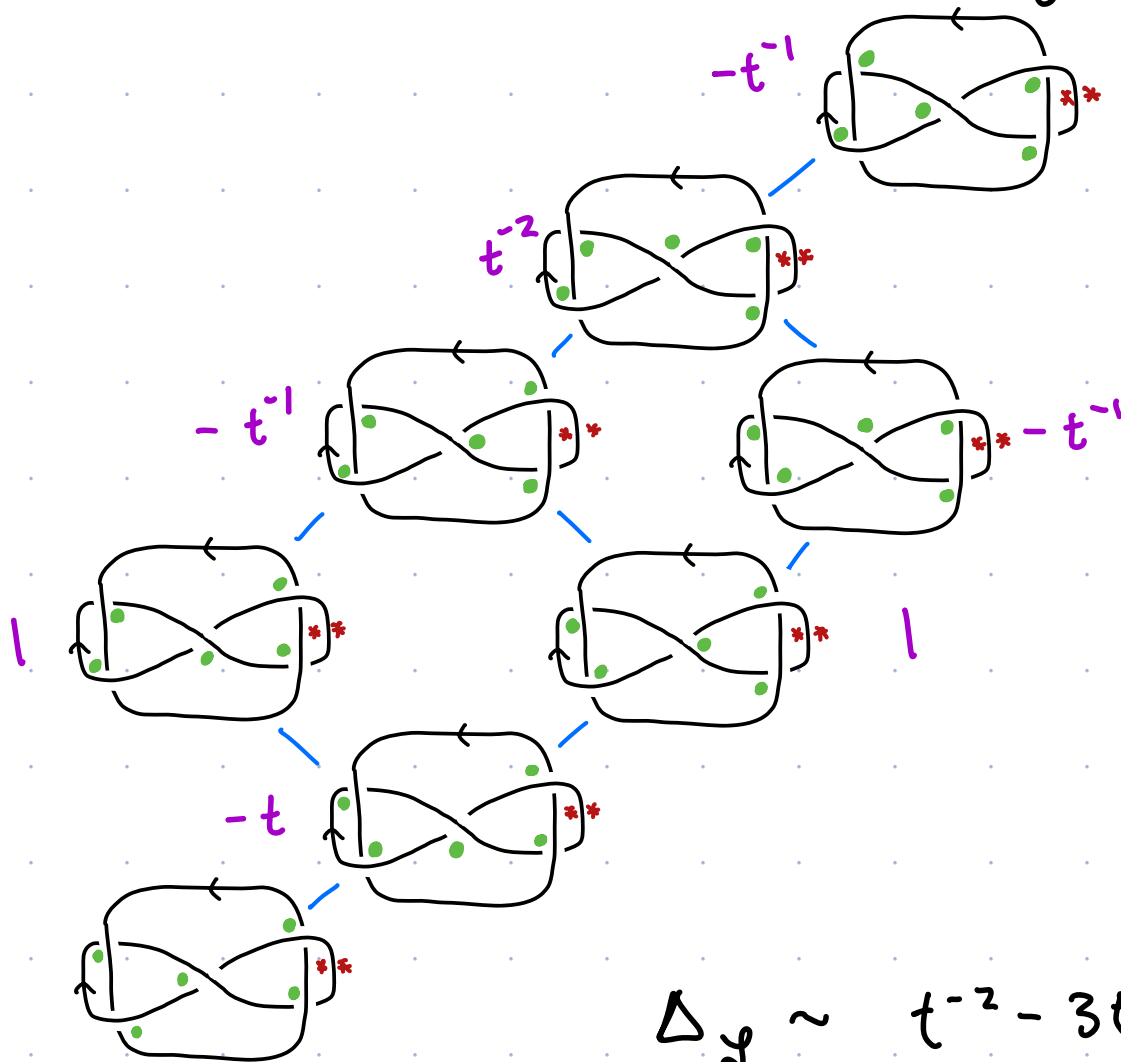
Oriented link diagrams L, L' of link $\mathcal{L}$

- $\mathcal{L}$ has many diagrams!
- Link invariants help distinguish btw different links.
 - ↖ statistic, polynomial, cohomology, etc.

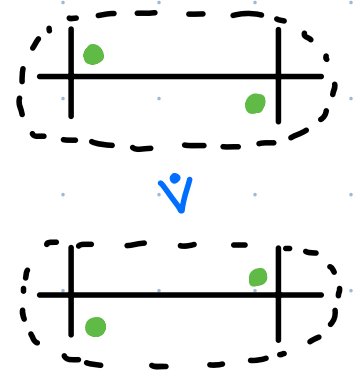
Alexander polynomial

- [Alexander 1923]: Alexander polynomial $\Delta_L \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$.
- [Kauffman 1983]: Can compute Δ_L using clock lattice

defined up to mult by $\pm t^n$ $n \in \mathbb{Z}$

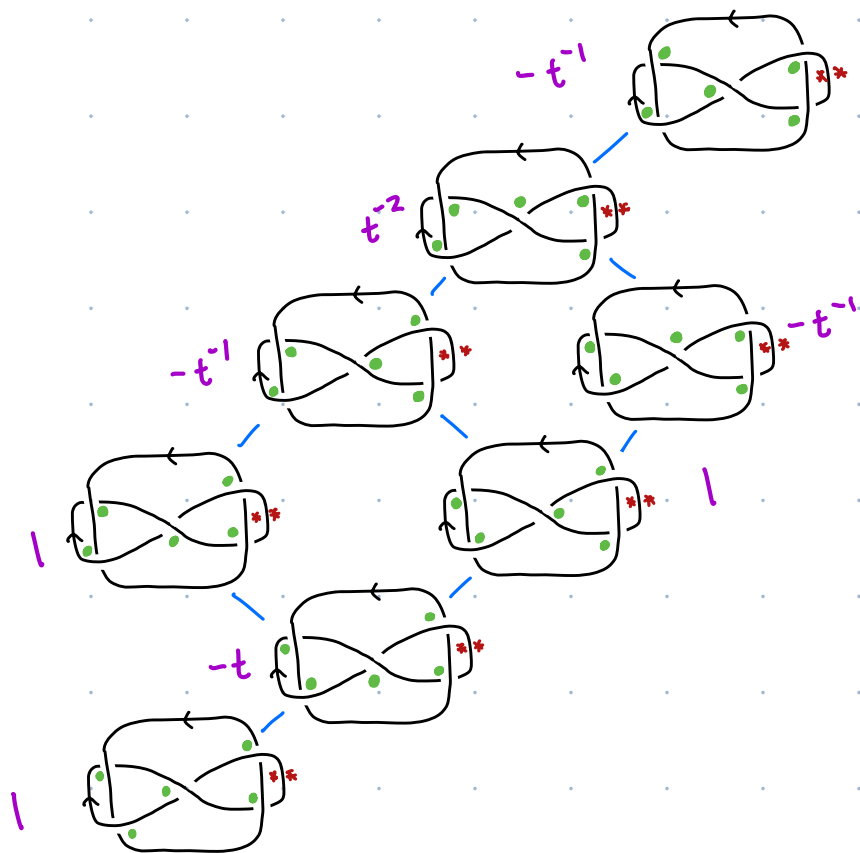


Cover relations



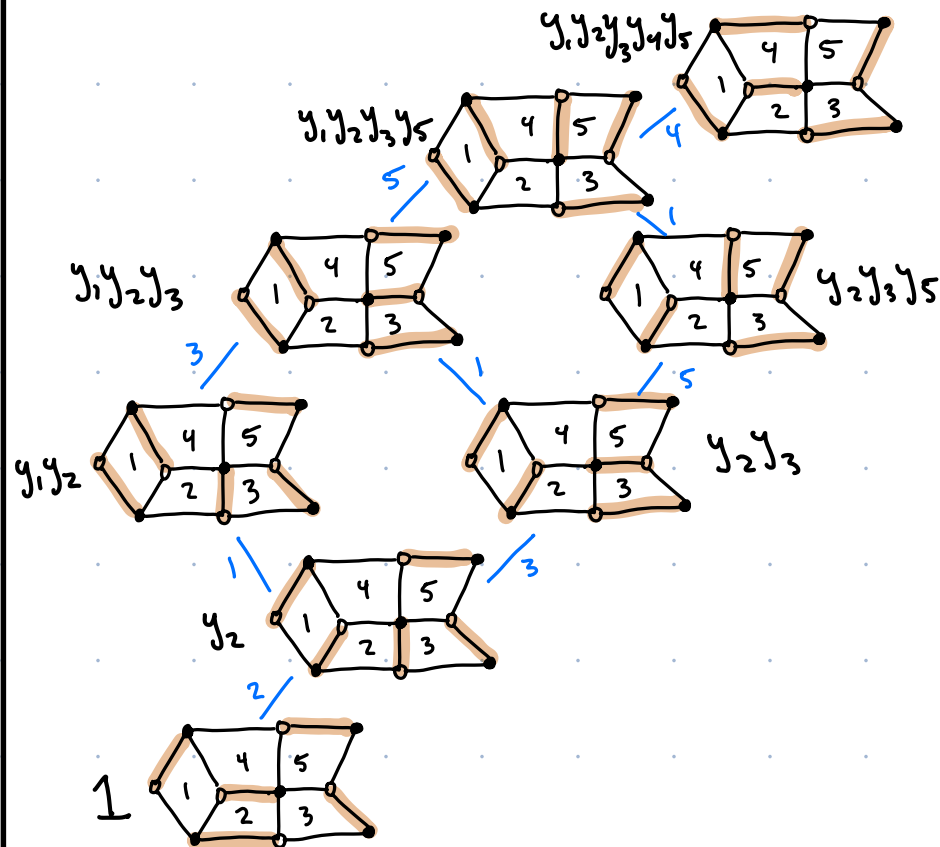
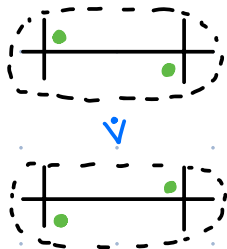
$$\Delta_L \sim t^{-2} - 3t^{-1} + 3 - t$$

Clock lattice vs. dimer lattice

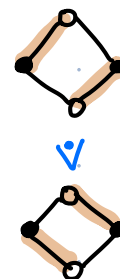


$$\Delta_{\mathcal{L}} \sim t^{-2} - 3t^{-1} + 3 - t$$

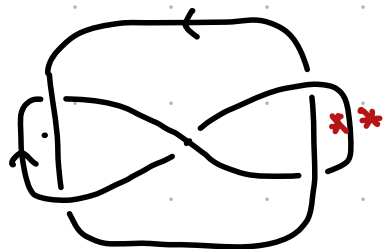
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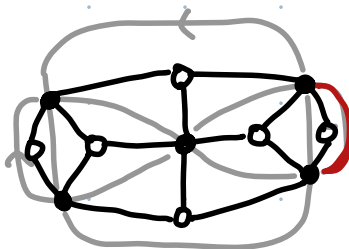
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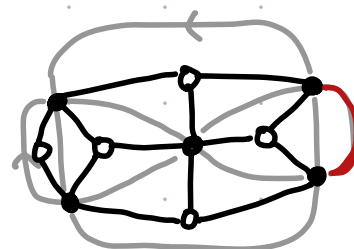
Specializing D_G to Δ_L



• on crossing
○ in (finite) face



delete ○
next to
special
segment



L oriented link diag.
(with choice of bdy segment)

G = face-crossing
incidence graph

G_L

Thm: [MMSBV] D_{G_L} is isomorphic to clock lattice of L .

Moreover, D_{G_L} specializes to Δ_L under

$$y_f \mapsto \begin{cases} -t & \text{if in face } f, \text{ see } \begin{array}{c} | \leftarrow | \\ | \end{array} \\ -t^{-1} & \text{if in face } f, \text{ see } \begin{array}{c} -| \leftarrow -| \\ | \end{array} \\ -1 & \text{else.} \end{cases}$$

Specializing D_G to $\Delta_{\mathcal{L}}$

Thm: [mmsBV] D_{G_L} is isomorphic to clock lattice of L .

Moreover, D_{G_L} specializes to $\Delta_{\mathcal{L}}$ under

$$y_f \mapsto \begin{cases} -t & \text{if in face } f, \text{ see } \begin{array}{c} | \leftarrow | - \\ | \end{array} \\ -t^{-1} & \text{if in face } f, \text{ see } \begin{array}{c} - | \leftarrow | \\ | \end{array} \\ 1 & \text{else.} \end{cases}$$

Notes: • [Cohen-Teicher '14] specialize Z_{G_L} to $\Delta_{\mathcal{L}}$

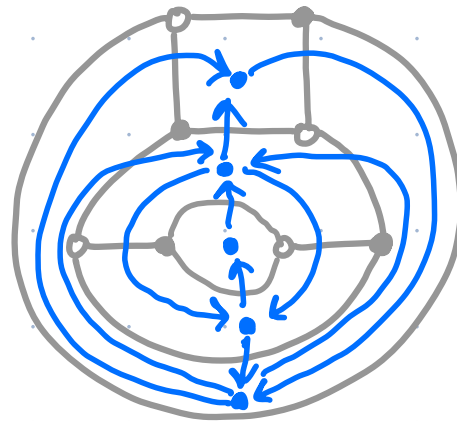
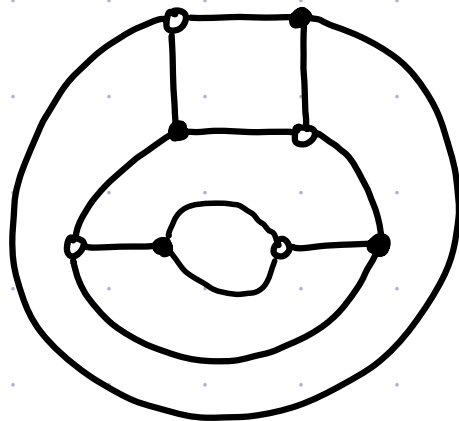
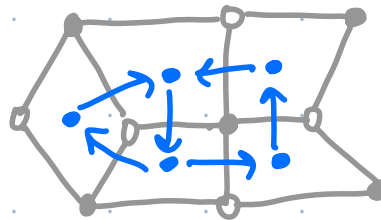
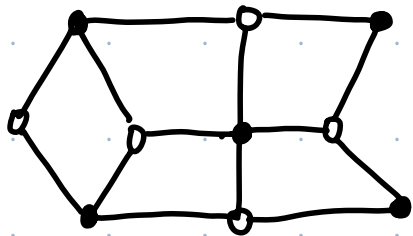
↑ partition fcn, edge variables

↳ use determinantal formulation of Z_{G_L} & $\Delta_{\mathcal{L}}$.

- Specialization of y_f inspired by [Bazier-Matte-Schiffler '21]
- (• Require that L is prime-like with no nugatory crossings)

D_G & cluster algebras

Thm: [mmsBV] D_G is an F-polynomial in the cluster algebra $A(Q_G)$. *→ Gives (many) recursive formulas for D_G in terms of $D_{G'}$*



G

Q_G

Cluster algebras

[Fomin-Zelevinsky '00]

Input: Q directed graph "quiver"

{ iterative combinatorial procedure
"mutation"
↓

Output: $A(Q)$ commutative ring w/ distinguished generators
"cluster algebra" "cluster variables"

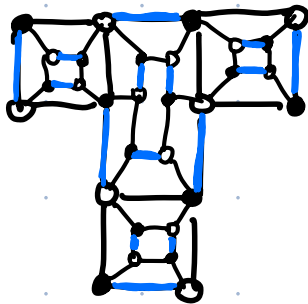
• F -polynomials $\leftrightarrow$ cluster variables (related by monomial substitution)

Complicated! Do not know non-recursive formula in general.

• Our result: If Q_G dual to $(*)$ -graph G , then (specific)
 F -poly is D_G . $\leftarrow A(Q_G)$ "knows" D_G .

Summary

- G a $(*)$ -graph $\rightsquigarrow D_G$ dimer lattice $\rightsquigarrow D_G$ dimer face polynomial
- If G comes from L link diagram, D_G generalizes Alexander polynomial $\Delta_L(t)$
- In general, D_G gives a special cluster variable/ F -polynomial in cluster algebra $A(Q_G)$. Induced subgraphs G' give rise to more.



thanks for listening!