Announcements

- M 11/28 from 3:10-4PM 204 Art
- W 11/30 6:10-8PM 6 Olson
- R 12/1 for 4:10-5PM 6 Wellman

MAT 21C REVIEW

10 Infinite Sequences and Series: 10.1-10.10

<table>
<thead>
<tr>
<th>Sequences</th>
<th>Partial Sums of</th>
<th>...Series</th>
<th>...Power Series</th>
<th>...Taylor’s Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>$s_0 = a_0$</td>
<td>$a_0$</td>
<td>$a_0$</td>
<td>$f(0)$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$s_1 = a_0 + a_1$</td>
<td>$a_0 + a_1x$</td>
<td>$f(0) + f'(0)x$</td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>$s_2 = a_0 + a_1 + a_2$</td>
<td>$a_0 + a_1x + a_2x^2$</td>
<td>$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$</td>
<td></td>
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<tr>
<td>...</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$a_n$</td>
<td>$s_n = \sum_{k=0}^{n} a_k$</td>
<td>$\sum_{k=0}^{n} a_kx^k$</td>
<td>$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$</td>
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Important Concepts

- limits
- convergence
- intervals of convergence

12 Vectors and the Geometry of Space: 12.1-12.5

Vectors

- A vector is an ordered set of real numbers: $\vec{v} = (v_1, v_2)$ or $\vec{v} = (v_1, v_2, v_3)$
- The length or magnitude of $\vec{v}$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$ or $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- If $\vec{v} \neq \vec{0}$, the direction of $\vec{v}$ is the unit vector $\frac{\vec{v}}{|\vec{v}|}$.
- $\vec{v} = |\vec{v}| \left( \frac{\vec{v}}{|\vec{v}|} \right)$

Vector Operations

For vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ and constant $k$

- Addition: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$
- Scalar Multiplication: $k \vec{u} = (ku_1, ku_2, ku_3)$
- Dot Product: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ (\(\rightleftharpoonup\) a scalar!)
- Cross Product: $\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1)$ (\(\rightleftharpoonup\) only in dimension 3!)

1
Properties of Vector Operations

- \( \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \), where \( \theta \) is the acute angle between \( \vec{u} \) and \( \vec{v} \)
- \( \vec{u} \cdot \vec{u} = |\vec{u}|^2 \)
- \( \vec{u} \) and \( \vec{v} \) are orthogonal when \( \vec{u} \cdot \vec{v} = 0 \)
- The vector projection of \( \vec{u} \) onto \( \vec{v} \) is \( \text{proj}_{\vec{v}} \vec{u} = \left( |\vec{u}| \cos \theta \right) \frac{\vec{v}}{|\vec{v}|} = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \)
- The scalar component of \( \vec{u} \) in the direction of \( \vec{v} \) is \( |\vec{u}| \cos \theta = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \right) \vec{v} = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \)
- \( \vec{u} \times \vec{v} = |\vec{u}| |\vec{v}| \sin \theta \hat{n} \), where \( \hat{n} \) is the unit vector pointing in the normal direction
- \( |\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta \) is the area of the parallelogram determined by \( \vec{u} \) and \( \vec{v} \)
- Nonzero vectors \( \vec{u} \) and \( \vec{v} \) are parallel when \( \vec{u} \times \vec{v} = \vec{0} \)
- Nonzero vectors \( \vec{u} \) and \( \vec{v} \) are parallel when \( \vec{u} = k \vec{v} \) for some scalar \( k \neq 0 \)
- \( \vec{u} \times \vec{v} = -(\vec{v} \times \vec{u}) \)

Lines and Planes in Space

- A vector equation for the line through \( P_0(x_0, y_0, z_0) \) parallel to \( \vec{v} \) is \( \vec{r}(t) = \vec{r}_0 + t \vec{v} \), where \( \vec{r}_0 = (x_0, y_0, z_0) \) is the position vector of \( P_0 \).
- A vector equation for the plane through \( P_0(x_0, y_0, z_0) \) normal to \( \vec{v} = (v_1, v_2, v_3) \) is
  \[ \vec{v} \cdot \overrightarrow{P_0P} = 0 \]
  or
  \[ v_1(x - x_0) + v_2(y - y_0) + v_3(z - z_0) = 0 \]
- The distance from a point \( S \) to a line through \( P \) parallel to \( \vec{v} \) is \( d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} \)
- The distance from a point \( S \) to a plane through \( P \) with normal \( \vec{v} \) is \( d = \frac{|\overrightarrow{PS} \cdot \vec{v}|}{|\vec{v}|} \)

13 Vector-Valued Functions and Motion in Space: 13.1-13.2

Curves in Space

- We can describe a curve in space as a collection of the positions in space traced by a particle’s path.
- If \( \vec{r}(t) = (f(t), g(t), h(t)) \) is the position vector of a particle moving along a smooth curve in space, then \( \vec{v}(t) = \frac{d\vec{r}}{dt} \) is the particle’s velocity vector and \( \vec{a}(t) = \frac{d^2\vec{r}}{dt^2} \), when it exists, is the particle’s acceleration.

Differentiation Rules

- Product Rule: \( \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \vec{v}(t) + \vec{u}(t) \vec{v}'(t) \)
- Chain Rule: \( \frac{d}{dt} [\vec{u}(f(t))] = f'(t) \vec{u}'(f(t)) \)
- When \( |\vec{r}(t)| = c \) is constant, \( \vec{r} \cdot \vec{r}' = 0 \). 

2
Initial Value Problem

• A differentiable function \( \vec{R}(t) \) is an antiderivative of a vector function \( \vec{r} \) if \( \frac{d\vec{R}}{dt} = \vec{r} \). Then the indefinite integral of \( \vec{r}(t) \) is \( \int \vec{r}(t) \, dt = \vec{R}(t) + \vec{C} \), for some constant vector \( \vec{C} \).

Differential Equation: \( \frac{d}{dt} \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \)

Initial Condition: \( \vec{r}(0) = \vec{r}_0 = \langle u_1, u_2, u_3 \rangle \)

Problem: Find \( \vec{r}(t) \).

1. Integrate differential equation:
   \[
   \vec{r}(t) = \int \frac{d}{dt} \vec{r}(t) \, dt = \left< \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right>
   = \langle F(t) + c_1, G(t) + c_2, H(t) + c_3 \rangle = \langle F(t), G(t), H(t) \rangle + \langle c_1, c_2, c_3 \rangle
   
   where \( F'(t) = f(t), G'(t) = g(t), H'(t) = h(t) \).

2. Use initial condition to solve for \( \vec{C} = \langle c_1, c_2, c_3 \rangle \):
   \[
   \langle u_1, u_2, u_3 \rangle = \vec{r}_0 = \langle F(0), G(0), H(0) \rangle + \langle c_1, c_2, c_3 \rangle
   
   u_1 = F(0) + c_1 \quad \Rightarrow \quad c_1 = u_1 - F(0)
   
   u_2 = G(0) + c_2 \quad \Rightarrow \quad c_2 = u_2 - G(0)
   
   u_3 = H(0) + c_3 \quad \Rightarrow \quad c_3 = u_3 - H(0)
   
3. Solution: \( \vec{r}(t) = \langle F(t) + u_1 - F(0), G(t) + u_2 - G(0), H(t) + u_3 - H(0) \rangle \)

14 Partial Derivatives: 14.1-14.8

Partial Derivatives

• The partial derivative of \( f(x,y) \) with respect to \( x \) at the point \( (x_0, y_0) \) is
  \[
  \frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)} = \frac{d}{dx} f(x,y_0) \bigg|_{x=x_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}
  
  \]

• \( f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \)

• Chain Rule: \( \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \)

Implicit Differentiation

Suppose \( f(x,y) = 0 \) and it is difficult to write \( y \) as a function of \( x \).

Problem: Find \( \frac{dy}{dx} \).

1. Make sure to find \( f \) in the form \( f(x,y) = 0 \). (Everything has to be on same side of the equal sign.)
2. Compute \( f_x, f_y \). Make sure \( f_y \neq 0 \).
3. Solution: \( \frac{dy}{dx} = -\frac{f_x}{f_y} \)
Directional Derivative and Gradient

- The directional derivative of \( f \) at \( P_0(x_0, y_0) \) in the direction of the unit vector \( \vec{u} = \langle u_1, u_2 \rangle \) is
  \[
  (D_{\vec{u}} f)_{P_0} = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}
  \]
powered{the limit exists.}
- The gradient vector of \( f(x, y) \) at a point \( P_0(x_0, y_0) \) is the vector \( \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \)
- \( D_{\vec{u}} f = \nabla f \cdot \vec{u} \)
- The function \( f \) increases most rapidly in the direction of \( \nabla f \).
- The derivative along a path is \( \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \)
- If a differentiable function \( f(x, y) \) has a constant value \( c \) along a smooth curve \( \vec{r} = x(t) \vec{i} + y(t) \vec{j} \), then \( f(x(t), y(t)) = c \). Here \( \vec{r}(t) \) describes a level curve of \( f \). Then \( \nabla f \) is orthogonal to the tangent vector \( \frac{d}{dt} \vec{r} \). The tangent line to a level curve \( f(x, y) = c \) at \( P_0(x_0, y_0) \) is
  \[
  f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0
  \]
- The tangent plane at the point \( P_0(x_0, y_0, z_0) \) on the level surface \( f(x, y, z) = c \) of a differentiable function \( f \) is the plane through \( P_0 \) normal to \( \nabla f \mid_{P_0} \). The tangent plane to a level surface \( f(x, y, z) = c \) at \( P_0(x_0, y_0, z_0) \) is
  \[
  f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0
  \]
The normal line of the surface at \( P_0 \) is the line through \( P_0 \) parallel to \( \nabla f \mid_{P_0} \). The normal line to a level surface \( f(x, y, z) = c \) at \( P_0(x_0, y_0, z_0) \) is
  \[
  x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t
  \]
- The linearization of a function \( f(x, y) \) at a point \( (x_0, y_0) \) where \( f \) is differentiable is the function
  \[
  L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
  \]

Optimization

- Suppose that \( f(x, y) \) and its first and second partial derivatives are continuous throughout a disk centered at \( (a, b) \) and that \( f_x(a, b) = f_y(a, b) = 0 \). Then
  1. \( f \) has a local maximum at \( (a, b) \) if \( f_{xx} < 0 \) and \( f_{xx}f_{yy} - f_{xy}^2 > 0 \).
  2. \( f \) has a local minimum at \( (a, b) \) if \( f_{xx} > 0 \) and \( f_{xx}f_{yy} - f_{xy}^2 > 0 \).
  3. \( f \) has a saddle point at \( (a, b) \) if \( f_{xx}f_{yy} - f_{xy}^2 < 0 \).
  4. the test is inconclusive at \( (a, b) \) if \( f_{xx}f_{yy} - f_{xy}^2 = 0 \). In this case, we must find some other way to
determine the behavior of \( f \) at \( (a, b) \).
- The method of Lagrange multipliers

\[
\text{Suppose that } f(x, y, z) \text{ and } g(x, y, z) \text{ are differentiable and } \nabla g \neq 0 \text{ when } g(x, y, z) = 0. \\

\text{Problem: Find local min/max values of } f \text{ subject to the constraint } g(x, y, z) = 0 \text{ (if they exist).}
\]

Solution: Find values of \( x, y, z \) and \( \lambda \) that simultaneously satisfy the equations
\[
\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0
\]
Lagrange multipliers with two constraints

Problem: Find extreme value of $f(x, y, z)$ whose variables are subject to two constraints

$$g_1(x, y, z) = 0 \quad g_2(x, y, z) = 0$$

where $g_1, g_2$ are differentiable with $\nabla g_1$ not parallel to $\nabla g_2$.

Solution: Locate points $P(x, y, z)$ where $f$ takes on its constrained extreme values. Find $x, y, z, \lambda, \mu$ that simultaneously satisfy

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \quad g_1(x, y, z) = 0 \quad \text{and} \quad g_2(x, y, z) = 0$$