Announcements

- HW3
- Q1R
- Midterm 1 Review

4 Exponential and Logarithmic Functions

\[ a^x \quad \log_a x \]
\[ e^x \quad \ln x \]
\[ a^x a^y = a^{x+y} \quad \ln xy = \ln x + \ln y \]
\[ (a^x)^y = a^{xy} \quad \ln x^n = n \ln x \]
\[ e^{\ln x} = x \quad \ln e^x = x \]

The natural exponential function is

\[
f(x) = e^x \quad \left( = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \exp(x) \right)
\]

Change of Base Formula

\[
\log_a x = \frac{\ln x}{\ln a}
\]

DO NOT ATTEMPT!

- \((a + b)^x \neq a^x + b^x\)
- \(\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}\)
- \(\frac{a^x}{a^x + b} \neq \frac{1}{1 + b}\)
- \(a^x = a^y \to x = y \quad a^x + a^y = a^z \not\to x + y = z\)
- \(\ln(x + y) \neq \ln x + \ln y\)
- \(\ln(x^n) \neq (\ln x)^n\)

Derivative of Exponential and Logarithmic Functions

Let \(u\) be a differentiable function of \(x\).

\[
\frac{d}{dx} [e^x] = e^x \\
\frac{d}{dx} [e^u] = e^u \frac{du}{dx} \\
\frac{d}{dx} [a^x] = (\ln a) a^x \\
\frac{d}{dx} [a^u] = (\ln a) a^u \frac{du}{dx} \\
\frac{d}{dx} [\ln x] = \frac{1}{x} \\
\frac{d}{dx} [\ln u] = \frac{1}{u} \frac{du}{dx} \\
\frac{d}{dx} [\log_a x] = \left(\frac{1}{\ln a}\right) \frac{1}{x} \\
\frac{d}{dx} [\log_a u] = \left(\frac{1}{\ln a}\right) \frac{1}{u} \frac{du}{dx}
\]
l’Hôpital’s Rule

Let \((a, b)\) be an interval that contains \(c\). Let \(f\) and \(g\) be differentiable in \((a, b)\), except possibly at \(c\).

When \(x = c\) directly substituted into \(\frac{f(x)}{g(x)}\) produces \(0/0\) or \(\pm\infty/\pm\infty\), then we say that the limit of \(\frac{f(x)}{g(x)}\) as \(x\) approaches \(c\) produces an indeterminate form.

**Rule.** If the limit of \(\frac{f(x)}{g(x)}\) as \(x\) approaches \(c\) produces an indeterminate form, then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]

provided the limit exists or is infinite.

**Rule.** If the limit of \(\frac{f(x)}{g(x)}\) as \(x\) approaches \(\infty\) produces an indeterminate form, then

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}
\]

provided the limit exists or is infinite. The same applies when \(x\) approaches \(-\infty\).

- Remember that the rule is to take the derivatives of the numerator and denominator separately

\[
\lim \frac{f'(x)}{g'(x)} \quad \text{NOT} \quad \lim \left( \frac{f(x)}{g(x)} \right)'
\]

- Sometimes it is necessary to apply l’Hôpital’s Rule more than once to remove an indeterminate form.

- L’Hôpital’s Rule can be used to compare the rates of growth of two functions.

**Problems**

A. Finding slopes of tangent lines  
B. Finding min or max  
C. Finding points of inflection  
D. Implicit Differentiation

**Applications**

A. Modeling a population To model the growth of a population an exponential function is often used. One way to model the growth of a populations whose quantity is restricted is called the logistic growth model

\[
f(t) = \frac{a}{1 + be^{-kt}}, \quad t \geq 0
\]

B. Compounding interest

An amount of \(P\), the principal, is deposited into an account at an annual interest rate of \(r\). The balance \(A_n\) after \(t\) years is

\[
A_n = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

where \(n\) is the number of compoundings in a year. The limit of \(A_n\) as \(n \to \infty\) is the balance \(A\) after \(t\) years of continuous compounding, or

\[
A = \lim_{n \to \infty} A_n = \lim_{n \to \infty} P \left( 1 + \frac{r}{n} \right)^{nt} = Pe^{rt}
\]