The Eynard-Orantin Recursion and Quantum Invariants in Geometry

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Motivation $=$ What I think I'm learning from physicists about "Quantization" and "Quantum" Something, and what they are speculating.

Quantum Objects

- Quantum Riemann Surface
- Quantum Invariants
- Quantization as A Mathematical Induction Formula

Quantum Objects: What they are

- Quantum Riemann Surface $=$ 2D Holonomic System
- Quantum Invariants $=$ Representation Theory of Quantum Algebras
- Quantum Algebras $=$ Deformations of Usual Algebras

Common feature $=$ Non-commutative deformations

When two completely different "quantum" things turn out to be exactly the same, something deep is happening.

More Concretely...

- Quantum Riemann Surface $=$ Differential Operator P
- Quantum Invariants $\Longleftrightarrow$ Generating Function Z

When physicists say these are the same thing

$$\text{Quantum Riemann Surface} \;=\; \text{Quantum Invariants},$$

they mean a **Schrödinger Equation**

$$PZ = 0.$$

Here P characterizes Z, and conversely, Z determines P.

Conjectural Example: Knot Invariants

Conjecture of Dijkgraaf, Fuji, Gukov, Sułkowski, ...

$$\begin{array}{ccc} \text{Knot A-polynomial } A_K(x,y) & = & \text{Classical invariant} \\ \text{Mirror Symmetry??}\big\downarrow & & \text{Eynard-Orantin Recursion}\big\downarrow \\ \text{Colored Jones polynomial } J_K(N,q) & = & \text{Quantum invariant} \end{array}$$

$$J_K(N,q) = Z(x,\hbar) \qquad \begin{cases} \hbar = 1/N \\ x = q^N \end{cases}$$

is the partition function of the EO theory. The Schrödinger equation is known as the AJ-conjecture of Garoufalidis (related to the hyperbolic volume conjecture):

$$\widehat{A}_K\left(x, \hbar\frac{d}{dx}\right) Z(x,\hbar) = 0.$$

The Eynard-Orantin Theory $\in$ The Analysis of Large N Matrix Models

The Input Data $=$ The Lagrangian Immersion

Consider a Lagrangian immersion:

$$\begin{array}{ccc} \iota : \Sigma & \longrightarrow & T^*\mathbb{C} \\ & & \Big\downarrow \pi \\ & & \mathbb{C} \end{array}$$

- $\Sigma =$ an open Riemann surface called the **Spectral Curve**;
- $T^*\mathbb{C} =$ the cotangent bundle of the complex line $\mathbb{C}$ (or $\mathbb{C}^*$);
- $\eta = ydx =$ the tautological (fundamental) 1-form on $T^*\mathbb{C}$ that determines the symplectic form $\omega = -d\eta$.

Lagrangian Singularities

We assume that the Lagrangian immersion

$$\begin{array}{ccc} \iota : \Sigma & \longrightarrow & T^*\mathbb{C} \\ & & \Big\downarrow \pi \\ & & \mathbb{C} \end{array}$$

has only **simple** Lagrangian singularities.

- A point $p \in \Sigma$ is a **Lagrangian singularity** if $d(\pi \circ \iota)(p) = 0$.
- The projection image $\pi \circ \iota(p)$ of a Lagrangian singularity is called a **caustic**.
- We assume that π is simply ramified at each caustic.

  $R = \{p_1, \dots, p_r\} =$ the set of Lagrangian singularities.

The Cauchy Kernel $= d\log$ (Riemann Prime Form)

Let $E(z_1, z_2)$ be the *Riemann prime form* on $\Sigma \times \Sigma$, and $\omega^{a-b}(z) = d\log E(z,a) - d\log E(z,b)$ the Cauchy integration kernel.

$$\begin{tikzcd} & \Sigma \times \Sigma \arrow[dl, "\alpha"'] \arrow[dr, "\beta=\mathrm{Res}"] & \\ \Sigma & & \Sigma \end{tikzcd}$$

For a meromorphic function $f(z)$ on Σ, we have

$$f(z) \longmapsto \beta_*(\omega^{a-b}(z) \cdot \alpha^* f(z)) = f(a) - f(b).$$

Here $\beta = \mathrm{Res}$ is the residue map with respect to the simple poles of $\omega^{a-b}(z)$ at $z = a$ and $z = b$.

The Recursion Kernel

At each Lagrangian singularity p_α, $\pi : \iota(\Sigma) \longrightarrow \mathbb{C}$ is locally a simply ramified double covering. Let

$$s_\alpha : \Sigma \ni z \longmapsto \tilde{z} \in \Sigma$$

denote the local Galois conjugate.

Now define the **Recursion Kernel** for z near p_α by

$$K_\alpha(z, z_1) = -\frac{1}{2}\,\frac{\omega^{\tilde{z}-z}(z_1)}{\eta(\tilde{z}) - \eta(z)}.$$

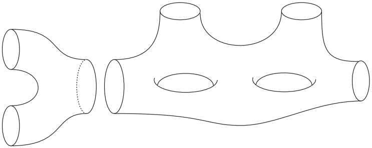
The Eynard-Orantin Topological Recursion (2007)

$$
\begin{aligned}
&W_{0,1}(z_1) = \iota^*\eta(z_1)\\
&W_{0,2}(z_1,z_2) = d_1 d_2 \log E(z_1,z_2) - \pi^* d_1 d_2 \log(x_1 - x_2)\\
&W_{g,n}(z_1,\dots,z_n) = \sum_{\alpha=1}^{r} \frac{1}{2\pi i} \oint_{\gamma_\alpha} K_\alpha(z,z_1)\\
&\quad \times \Bigg[W_{g-1,n+1}(z,\tilde{z},z_2,\dots,z_n)\\
&\quad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{no }(0,1)-\text{term}} W_{g_1,|I|+1}(z,z_I) W_{g_2,|J|+1}(\tilde{z},z_J)\Bigg].
\end{aligned}
$$

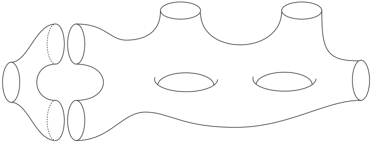
The origin: $W_{g,n} =$ the $(1/N)^{2g-2+n}$ term of the n-point correlation function of resolvents of random matrices of size N.

Topological Recursion = Degeneration on $\overline{\mathcal{M}}_{g,n}$, Or what topologists say *removal of a pair-of-pants*.

$$(g, n) \Longrightarrow (g, n-1)$$

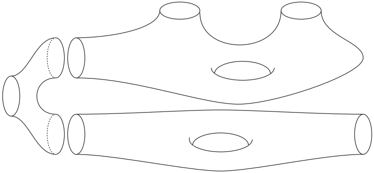


$$(g, n) \Longrightarrow (g-1, n+1)$$



$$(g, n) \Longrightarrow (g_1, n_1+1)+(g_2, n_2+1)$$

$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 - 1 \end{cases}$$



Many equations in QFT are mathematically the same thing.

The Eynard-Orantin Recursion is a manifestation of the equivalence:

$$
\begin{tikzcd}
 & \text{Loop Equation} \arrow[d, "\text{Correlation of resolvents}"', "\text{RMT}"] \\
\text{Ward-Takahashi Identity} \arrow[r, "BCOV", "DV"'] & \text{Eynard-Orantin Recursion} \\
 & \text{Schwinger-Dyson Equation} \arrow[u, "\text{Genus Expansion}"]
\end{tikzcd}
$$

What is the EO recursion good for?

The Eynard-Orantin Recursion is a universally applicable formula that **conjecturally** computes

1. Intersection numbers of $\overline{\mathcal{M}}_{g,n}$. (Proved)
2. Hurwitz numbers and their generalizations. (Proved)
3. Gromov-Witten Invariants.
4. Quantum Knot Invariants, such as Jones Polynomials and HOMFLY polynomials.

By choosing different Lagrangian immersions we can calculate a variety of quantum topological invariants.

Most Notably: the Remodeling Conjecture

Conjecture (Mariño 2006, Bouchard-Klemm-Mariño-Pasquetti 2007.)

- $X =$ toric Calabi-Yau 3-fold
- $\Sigma =$ the mirror curve of X
- $\Longrightarrow W_{g,n}$ is the generating function, or the **Laplace transform**, of *open* Gromov-Witten invariants of X.

Proof for the case of non-singular varieties proposed in Eynard-Orantin 2012.

The General Picture

Mirror Symmetry $=$ Laplace Transform

Conjectural Relations to Differential Equations, Part 1

Free Energies

A symmetric function $F_{g,n}$ on Σ^n is a **Free Energy** of type (g,n) if

$$W_{g,n}(z_1,\ldots,z_n) = d_1\cdots d_n F_{g,n}(z_1,\ldots,z_n).$$

Partition Function

Define the **Partition Function** by

$$Z(z_1,z_2,\ldots;\hbar) = \exp\left(\sum_{g,n}\frac{1}{n!}\hbar^{2g-2+n}F_{g,n}(z_1,\ldots,z_n)\right)\cdot\vartheta.$$

Conjecture (Eynard-Orantin 2007, Borot-Eynard 2011)

There is a systematics choice of the free energies so that

$$Z(z_1, z_2, \ldots; \hbar) = \exp\left(\sum_{g,n} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}(z_1, \ldots, z_n)\right) \cdot \vartheta$$

is a τ-function of a KP (KdV) type integrable system of PDEs, *after a suitable modification with a theta function factor*.

Conjectural Relations to Differential Equations, Part 2

Define the **diagonal value of the Partition Function** by

$$Z(z,\hbar) = \exp\left(\sum_{g,n} \frac{1}{n!}\hbar^{2g-2+n} F_{g,n}(z,z,\ldots,z)\right)\cdot\vartheta(z).$$

Since $F_{g,n}$ is a symmetric function, this substitution is what is known as the **Principal Specialization.**

Conjecture on Holonomic Systems–Schrödinger Equation

The principal specialization satisfies Schrödinger Equation

$$P\left(x,\hbar\frac{d}{dx}\right)\, Z\big(z(x),\hbar\big) = 0,$$

where $z = z(x)$ is considered as a multivalued function, and the equation holds on each branch of the coving $\pi : \Sigma \longrightarrow \mathbb{C}$.

Spectral Curve as the Characteristic Variety

Conjecture on the Characteristic Variety

Moreover, the characteristic variety of the Schrödinger operator $P\left(x, \hbar \frac{d}{dx}\right)$ coincides with the Lagrangian immersion:

$$\begin{array}{ccc} \iota : \Sigma & \longrightarrow & T^*\mathbb{C} \\ & & \Big\downarrow \pi \\ & & \mathbb{C} \end{array}$$

In other words, the spectral curve is the characteristic variety of the Schrödinger equation:

$$\Sigma = \{(x, y) \in T^*\mathbb{C} \mid P(x, y) = 0\}.$$

We wish to have a few simple mathematical examples that exhibit all important features of this structure, for which we can compute everything, and from which we can learn what's going on.

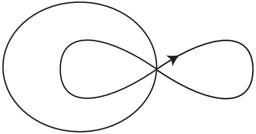
Simple examples come from the **Laplace Transform** of

1. **Higher-genus Catalan numbers**, and
2. Various variants of **Hurwitz numbers**.

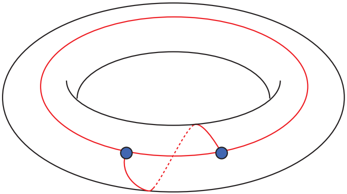
Cellular Graph

Cellular graph $=$ the 1-skeleton of a cell-decomposition of a connected surface.

Genus 0 example:



Genus 1 example:



Counting Cellular Graphs

Let $D_{g,n}(\mu_1, \ldots, \mu_n)$ denote the automorphism-weighted count of the number of connected cellular graphs on an oriented surface of genus g with labeled n vertices of degrees $(\mu_1, \ldots, \mu_n)$.

Higher-genus Catalan Numbers

$$C_{g,n}(\mu_1, \ldots, \mu_n) = \mu_1 \cdots \mu_n D_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}.$$

The $(g, n) = (0, 1)$ case is the original Catalan numbers:

$$C_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m} = C_m = \dim \mathrm{End}_{\mathcal{U}_q(s\ell(2))}(T^{\otimes m}\mathbb{C}^2).$$

The Catalan Recursion [Dumitrescu-M-Safnuk-Sorkin]

The following equation holds:

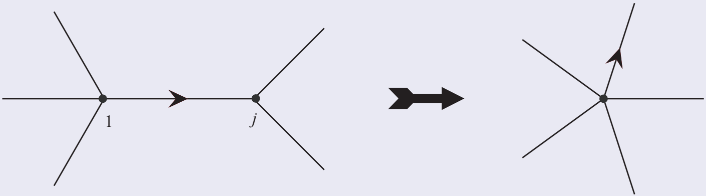
$$C_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{j=2}^{n} \mu_j C_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_n)$$
$$+ \sum_{\alpha+\beta=\mu_1-2} \Bigg[C_{g-1,n+1}(\alpha, \beta, \mu_2, \ldots, \mu_n)$$
$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}} C_{g_1,|I|+1}(\alpha, \mu_I)\, C_{g_2,|J|+1}(\beta, \mu_J) \Bigg].$$

The $(g, n) = (0, 1)$ case reduces to the classical formula

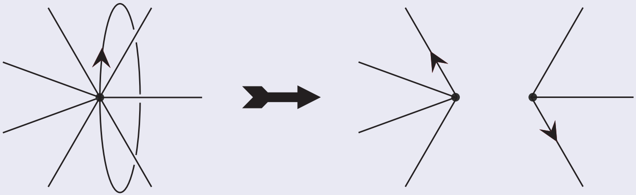
$$C_m = \sum_{a+b=m-1} C_a\, C_b.$$

Proof: Shrink an Edge

Case 1: The edge we try to shrink connects Vertex 1 to Vertex j.



Case 2: The edge is a loop attached to Vertex 1.



The Mirror Dual of the Catalan Numbers

Mirror Symmetric Dual of Catalan Numbers

Define

$$z(x) = \sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}}.$$

From the Catalan recursion we see

$$x = z + \frac{1}{z}.$$

Lagrangian immersion

$$x = z + \frac{1}{z} \quad \text{and} \quad y = -z$$

defines a Lagrangian immersion.

The Laplace Transform

Define the Laplace transform of $D_{g,n}(\vec{\mu})$ by

$$F_{g,n}(t_1,\dots,t_n) = \sum_{\vec{\mu}\in\mathbb{Z}_+^n} D_{g,n}(\vec{\mu})\, e^{-(\mu_1 w_1+\cdots+\mu_n w_n)},$$

where variables are related by

$$e^{w_i} = x_i = z_i + \frac{1}{z_i}, \qquad z_i = \frac{t_i+1}{t_i-1}.$$

Define also the Laplace transform of the Catalan numbers by

$$W_{g,n} = d_1 \cdots d_n F_{g,n}.$$

Theorem [Chapman-M-Safnuk, M-Penkava]

1. $W_{g,n}$ satisfies the Eynard-Orantin topological recursion, which is exactly the Laplace transform of the Catalan recursion.
2. For $2g-2+n>0$,

$$F_{g,n}(t_1,\ldots,t_n) = F_{g,n}(1/t_1,\ldots,1/t_n)$$

is a Laurent polynomial of degree $6g-6+3n$.

3. The principal specialization $F_{g,n}(t,t,\ldots,t)$ is the virtual Poincaré polynomial of $\mathcal{M}_{g,n}$ in the variable $\xi = \frac{(t+1)^2}{4t}$.
4. In particular,

$$F_{g,n}(1,1,\ldots,1) = (-1)^n \chi(\mathcal{M}_{g,n}).$$

Theorem [Chapman-M-Safnuk, M-Penkava] Continued

The top degree terms of $F_{g,n}$ know the intersection theory on $\overline{\mathcal{M}}_{g,n}$:

$$F^{\mathrm{top}}_{g,n}(t_1,\ldots,t_n) = \frac{(-1)^n}{2^{2g-2+n}} \sum_{k_1,\ldots,k_n} \langle \tau_{k_1}\cdots\tau_{k_n}\rangle_{g,n} \prod_{i=1}^{n} (2k_i-1)!! \left(\frac{t_i}{2}\right)^{2k_i+1}.$$

The recursion restricts to the top degree terms, and gives a visual explanation of the Witten-Kontsevich theorem.

Moreover, the conjectural relations to differential equations also hold.

Observation [M-Sułkowski]

The diagonal partition function is a **MATRIX MODEL!!!**

$$Z(z,\hbar) = \exp\left(\sum_{g,n} \frac{1}{n!}\hbar^{2g-2+n} F_{g,n}(z,z,\ldots,z)\right)$$
$$= \int_{\mathcal{H}_{N\times N}} \det\left(1-\sqrt{-\xi}X\right)^N e^{-\frac{1}{2}N\cdot\mathrm{trace}(X^2)}.$$

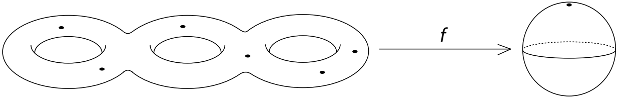
This is the **Principal Specialization** of a KP τ-function, and satisfies the **Schrödinger equation**

$$\left(\hbar^2\frac{d^2}{dx^2}+\hbar x\frac{d}{dx}+1\right) Z(z,\hbar)=0.$$

Here $\hbar = 1/N$ and $\xi = \frac{(t+1)^2}{4t} = \frac{z^2}{z^2-1}$. The characteristic variety of this operator is $x = z+\frac{1}{z}$.

Single Hurwitz Numbers

$H_{g,n}(\vec{\mu})$ = the automorphism-weighted count of meromorphic functions f on a connected Riemann surface of genus g with n labeled poles of orders $(\mu_1, \ldots, \mu_n)$, such that df has unlabeled simple zeros.



Theorem [Eynard-M-Safnuk, M-Zhang]

1. The Laplace transform of $H_{g,n}(\vec{\mu})$ satisfies the Eynard-Orantin recursion (the Bouchard-Marino conjecture), which is the Laplace transform of the **cut-and-join** equation.
2. The Lagrangian immersion is given by $x = e^u = ze^{-z}$ and $y = z$ with the tautological 1-form $\eta = y\frac{dx}{x}$.

Remark [M-Sułkowski]

The diagonal partition function $Z(z, \hbar)$ for the single Hurwitz numbers is the principal specialization $p_j = (x/\hbar)^j$ of a KP τ-function.

Theorem [J. Zhou]

The diagonal partition function satisfies

$$\left(\hbar \frac{d}{du} - e^u e^{\hbar \frac{d}{du}}\right) Z(z, \hbar) = 0.$$

The characteristic variety of this difference equation is the Lagrangian immersion $x = e^u = ze^{-z}$ and $y = z$ in $T^*\mathbb{C}^*$.

More General Orbifold Hurwitz Numbers

Theorem [M-Shadrin-Spitz]

1. The diagonal partition function of the Laplace transform of the orbifold Hurwitz numbers is the principal specialization of a KP τ-function.
2. It satisfies the Schrödinger-type equation, and the characteristic variety agrees with the spectral curve of the theory.

Orbifold Hurwitz Numbers [Bouchard, Hernández-Serrano, Liu, M]

1. Let $H_{g,n}(\vec{\mu}; r)$ be the automorphism-weighted count of meromorphic functions on a connected Riemann surface of genus g with n labeled poles of orders $(\mu_1, \ldots, \mu_n)$ and m unlabeled zeros of a fixed degree $r > 0$, such that all other zeros of df are simple.
2. Define the Laplace transform

$$F^{(r)}_{g,n}(x_1, \ldots, x_n) = \sum_{\vec{\mu} \in \mathbb{Z}^n_+} H_{g,n}(\vec{\mu}; r) e^{-(\mu_1 w_1 + \cdots + \mu_n w_n)}.$$

3. Then $W^{(r)}_{g,n} = d_1 \cdots d_n F^{(r)}_{g,n}$ satisfies the EO recursion with respect to the Lagrangian immersion

$$\begin{cases} x = e^{-w} = z e^{-z^r} \\ y = z^r. \end{cases}$$

Theorem [M-Shadrin-Spitz]

1. The diagonal partition function is given by

$$Z(x,\hbar)=\sum_{m=0}^{\infty}\frac{1}{r^m m!}\frac{x^{rm}}{\hbar^m}\exp\left[\hbar\,\frac{rm(rm-1)}{2}\right]$$

   from a KP τ-function via principal specialization $p_j=(x/\hbar)^j$.

2. The Schrödinger equation is given by

$$\left(\hbar\frac{d}{dw}+e^{-\frac{r-1}{2}\hbar\frac{d}{dw}}\cdot e^{-rw}\cdot e^{\frac{r-1}{2}\hbar\frac{d}{dw}}\cdot e^{-r\hbar\frac{d}{dw}}\right)Z(e^{-w},\hbar)=0.$$

3. The characteristic variety of the Schrödinger operator is

$$x=e^{-w}=ze^{-z^r}\quad\text{and}\quad y=z^r.$$

Conclusion. Mirror Symmetry = Laplace Transform

- The analysis technique of RMT is extremely useful in computing **Quantum Topological Invariants**, i.e., solving enumerative geometry problems.
- The topological recursion of Eynard-Orantin, originally discovered as the loop equation for the correlation functions of resolvents of random matrices, is the **Laplace transform** of the combinatorial equation that comes from the pair-of-pants decomposition.
- Spectral curve of the EO theory is the mirror symmetric dual of the $g = 0, n = 1$ quantum topological invariant.
- The characteristic variety of the Schrödinger equation for the partition function is the Lagrangian immersion determined by the Laplace transform of the $g = 0, n = 1$ quantum topological invariant.

Reference

1. M.-Zhang: *Polynomial recursion formula for linear Hodge integrals*, Commun. Number Theory and Physics **4** (2010).
2. Chapman-M.-Safnuk: *The Kontsevich constants for the volume of the moduli of curves and topological recursion*, Commun. Number Theory and Physics **5** (2011).
3. Eynard-M.-Safnuk: *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, Publ. Res. Inst. Math. Sci. **47** (2011).
4. M.-Penkava: *Topological recursion for the Poincaré polynomial of the combinatorial moduli space of curves*, Advances in Mathematics **230** (2012).
5. Dumitrescu-M.-Safnuk-Sorkin: *The spectral curve of the Eynard-Orantin recursion via the Laplace transform*, AMS Contemporary Mathematics **593** (2013).

Forthcoming Papers

- Dumitrescu, Hernández Serrano, Kimura, and M.: *Topological recursion for Gromov-Witten invariants of the classifying space of a finite group*
- M. and Sułkowski: *Spectral curves and the Schrödinger equations for the Eynard-Orantin recursion*
- M., Shadrin, and Spitz: *The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures*
- Bouchard, Hernández Serrano, Liu and M.: *Mirror symmetry of orbifold Hurwitz numbers*

Happy 80-th Birthday, Harold!