New formulae for Gromov-Witten invariants and Hurwitz numbers

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UC Davis
March 2nd, 2009

The tale of an

unconventional tire maker

The raw material: a rubber tree

Manufacturing prcoess: creating objects with nice properties

Application (original motivation): wheels of a car

Thinking outside the box: building an earthship!

Outline: The tale of an unconventional geometer

1. The recursion
   - The raw material: Geometric data
   - The manufacturing process: The recursion
2. Applications
   - The original motivation: Random matrix theory
   - Thinking outside the box: Enumerative geometry

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Geometric data

Definition

A spectral curve C is a triple $(\mathcal{C}, x, y)$, where $\mathcal{C}$ is a genus $\bar{g}$ compact Riemann surface, and x and y are two holomorphic functions on some open set in $\mathcal{C}$. We assume that the ramification points of x have multiplicity 2.

Examples:

- $\mathcal{C} = \mathbb{C}_\infty = \mathbb{P}^1, \quad x(p) = p^2, \quad y(p) = p$
  $\to y^2 = x.$
- $\mathcal{C} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad x(p) = \wp(p; g_2, g_3), \quad y(p) = \wp'(p; g_2, g_3)$
  $\to y^2 = 4x^3 - g_2 x - g_3.$

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The Bergmann kernel

We choose a symplectic basis of cycles (A_i, B_j) on the genus $\bar{g}$ Riemann surface $\mathcal{C}$, such that

$$A_i \cap B_j = \delta_{i,j}, \qquad i,j = 1, \ldots, \bar{g}.$$

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The Bergmann kernel $B(p_1, p_2)$ is the unique bilinear differential on $\mathcal{C}$ having a double pole at $p_1 = p_2$ with no residue, no other poles, and such that, in good local coordinates,

$$B(p_1, p_2) \underset{p_1 \to p_2}{\sim} \frac{\mathrm{d}p_1 \mathrm{d}p_2}{(p_1 - p_2)^2} + \text{reg}, \qquad \oint_{A_i} B(p_1, p_2) = 0 \quad \forall i.$$

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The recursion

Definition: the recursion

Given a spectral curve $C = (\mathcal{C}, x, y)$, we define the meromorphic forms:

$$\omega_1^{(0)}(p_1) = y(p_1)\mathrm{d}x(p_1), \qquad \omega_2^{(0)}(p_1, p_2) = B(p_1, p_2),$$

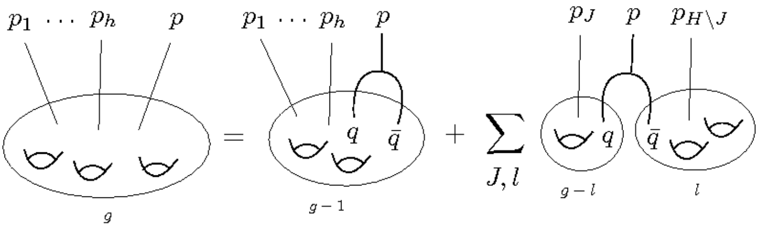
and using H as a collective notation $H = \{p_1, \ldots, p_h\}$, by recursion on g and h, for $2g - 2 + h \geq 0$,

$$\omega_{h+1}^{(g)}(p, H) = \sum_m \operatorname*{Res}_{q \to a_m} K_m(p, q) \Big[\omega_{h+2}^{(g-1)}(q, \bar{q}, H) \\ + \sum_{l=0}^{g} \sum_{J \subset H}{}' \omega_{1+|J|}^{(g-l)}(q, J) \omega_{1+h-|J|}^{(l)}(\bar{q}, H \backslash J) \Big],$$

where $\sum'$ means that we exclude the terms $(l, J) = (g, \emptyset), (0, H)$.

The recursion (pictorial representation)

$$\omega^{(g)}_{h+1}(p,H) = \sum_m \operatorname*{Res}_{q\to a_m} K_m(p,q)\Big[\omega^{(g-1)}_{h+2}(q,\bar q,H) + \sum_{l=0}^{g}\sum_{J\subset H}{}' \omega^{(g-l)}_{1+|J|}(q,J)\,\omega^{(l)}_{1+h-|J|}(\bar q, H\backslash J)\Big],$$



The invariants

Definition: the invariants

For $g \geq 2$, we define the invariants $F_g := \omega_0^{(g)}$ by

$$F_g = \frac{1}{2-2g} \sum_m \operatorname*{Res}_{q \to a_m} \omega_1^{(g)}(q) \Phi(q),$$

where $\mathrm{d}\Phi(q) = \omega_1^{(0)}(q) = y(q)\mathrm{d}x(q)$.

Some nice properties

The $\omega_h^{(g)}(p_1, \ldots, p_h)$ and F_g have many nice properties [EO]. Some examples:

- $\omega_h^{(g)}(p_1, \ldots, p_h)$ is a meromorphic one-form in each of its variable, with poles only at the ramification points, with order at most $6g - 6 + 2h + 2$, and vanishing residue;
- $\omega_h^{(g)}(p_1, \ldots, p_h)$ is symmetric in its h variables;
- Any two curves $C = (\mathcal{C}, x, y)$ and $\tilde{C} = (\tilde{\mathcal{C}}, \tilde{x}, \tilde{y})$ which are related by a conformal mapping $\mathcal{C} \to \tilde{\mathcal{C}}$ preserving the symplectic form $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$ have the same invariants F_g;
- More properties, related to integrable systems, ...

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Summary so far

1. Raw material: A spectral curve $C = (\mathcal{C}, x, y)$
2. Manufacturing process: From this raw material we construct an infinite tower of meromorphic forms $\omega_h^{(g)}(p_1, \ldots, p_h)$ and invariants F_g
3. Quality control testing: The invariants satisfy many nice properties.

Are these objects of any use?

What are they computing?

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Random matrix theory in a nutshell (I)

- 0-dimensional QFT, with basic field a $N \times N$ Hermitian matrix
- Free energy:

$$F = \log Z = \log \left(\frac{1}{\mathrm{vol}\,U(N)} \int \mathrm{d}M \mathrm{e}^{-\frac{1}{g_s} V(M)} \right),$$

  with $V(M)$ a polynomial potential
- F has a perturbative expansion (large N expansion)

$$F = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2} t^h := \sum_{g=0}^{\infty} F_g g_s^{2g-2},$$

  with t the 't Hooft parameter $t = N g_s$ (fatgraph expansion)
- Goal: solve for F and for the correlation functions

$$W_n(x_1, \ldots, x_n) = \langle \mathrm{Tr} \frac{1}{x_1 - M} \cdots \mathrm{Tr} \frac{1}{x_n - M} \rangle$$

  which also have a genus g expansion

Random matrix theory in a nutshell (II)

- The $W_{n,g}$ satisfy a set of differential equations, called loop equations. At genus 0, it becomes an algebraic equation:

$$(W_{1,0}(x))^2 = V'(x) W_{1,0}(x) - P_{1,0}(x),$$

with $P_{1,0}(x)$ some polynomial

- In terms of $y(x) = V'(x) - 2W_{1,0}(x)$, it defines a spectral curve

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Theorem [Eynard-Orantin]

The $\omega_h^{(g)}(p_1, \ldots, p_h)$ and F_g computed by the recursion applied to this spectral curve correspond respectively to the correlation functions and free energies of the matrix integral in the large N expansion.

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Different perspective

"Forget" about matrix integrals and spectral curves, and consider the recursion *per se*.

My goal:

- Convince you that for some choices of spectral curves, the $\omega_h^{(g)}(p_1, \ldots, p_h)$ and F_g have a completely different interpretation as generating functions of Gromov-Witten invariants and Hurwitz numbers.
  1. Describe the enumerative invariants
  2. Sketch the string theory framework underpinning the invariants, and how dualities may be used to compute generating functions
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Enumerative geometry

What is enumerative geometry?

The art of counting geometric structures satisfying a certain set of geometric conditions.

Examples:

- What is the number of intersection points of two lines in $\mathbb{P}^2$?
- Let X be a general quintic threefold (hypersurface of degree 5 in $\mathbb{P}^4$). How many lines are contained in X?

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More precisely (but still roughly):

- Generically, the moduli space of curves in X is not well behaved (not compact, ...)
- There exists a nice compactification, due to Kontsevich, which consists in considering the moduli space of (0-pointed) stable maps $f : \Sigma_g \to X$, where Σ_g is a genus g curve with $f_*(\Sigma_g) = \beta$. We denote this moduli space by $M_g(X, \beta)$.

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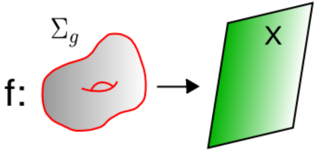

(Well I need a second one :-)

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- We can form the generating functions

$$F_g = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta N_{g,\beta},$$

with q a formal variable.

- You have now discovered A-model topological string theory!

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(Well I need a second one :-)

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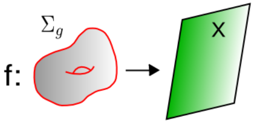
The A-model topological string theory

Roughly, topological string theory is a theory of maps from Riemann surfaces (string worldsheets) to a target space X, which must be a Calabi-Yau threefold.



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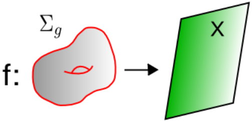

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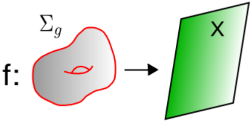

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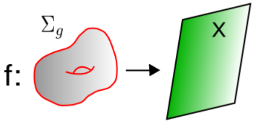

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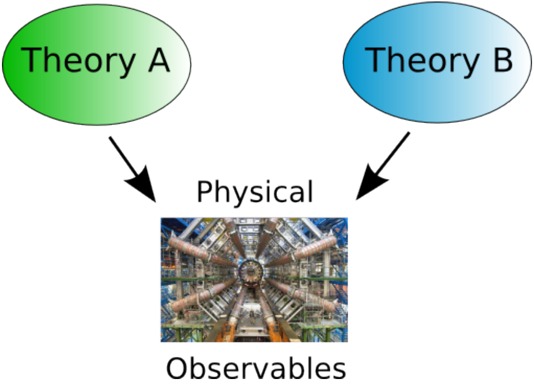

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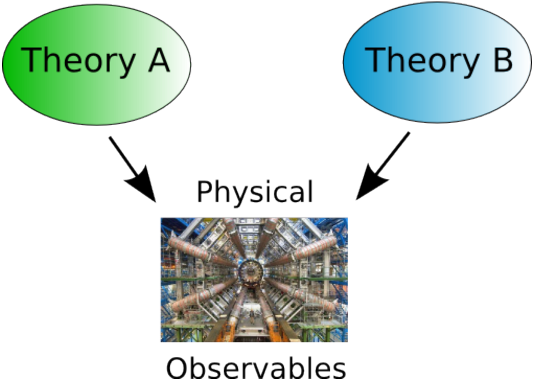

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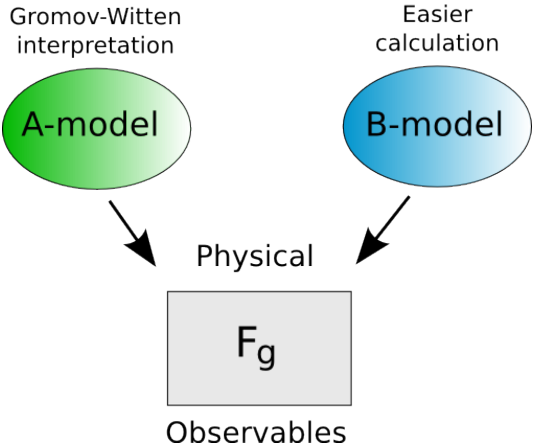

Toric Calabi-Yau threefolds

Consider the A-model on a (noncompact) toric Calabi-Yau threefold X (example: $X = K_{\mathbb{P}^2}$).

- The mirror is the B-model on a family of noncompact Calabi-Yau threefolds Y given by a hypersurface

$$\{ww' = H(x, y; t)\} \subset (\mathbb{C})^2 \times (\mathbb{C}^*)^2,$$

where $H(x, y; t)$ is a Laurent polynomial in $x, y \in \mathbb{C}^*$ [Hori-Vafa, Givental].

- The singular locus of the fibration over $(\mathbb{C}^*)^2$ is a family of curves Σ (mirror curve) given by

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Mirror B-model geometry

The threefold Y: $\{ww' = H(x, y; t)\} \subset (\mathbb{C})^2 \times (\mathbb{C}^*)^2$

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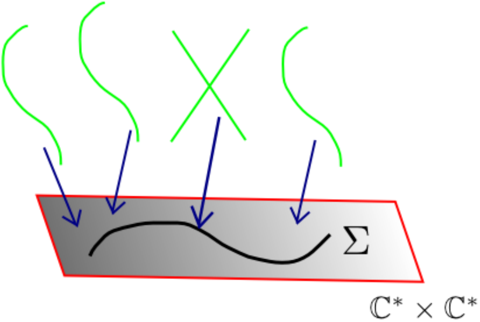

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1. Generating functions F_g of GW invariants of $X \leftrightarrow$ genus g free energies of A-model on X

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Our main conjectures

Conjecture I [BKMP]

Let C be the spectral curve defined by the singular locus of the noncompact Calabi-Yau threefold Y mirror to a toric Calabi-Yau threefold X. The F_g's computed by the recursion are the B-model genus g amplitudes, which are mapped by the mirror map to the generating functions of genus g Gromov-Witten invariants of X.

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The $\omega_n^{(g)}(p_1, \ldots, p_n)$ computed by the recursion are the open B-model genus g, h hole amplitudes, which are mapped by the mirror map to the generating functions of genus g, h hole open Gromov-Witten invariants of X.

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Evidence

- Computational evidence: in [BKMP], we computed the F_g and $\omega_n^{(g)}(p_1, \ldots, p_n)$ for many X and found perfect agreement with GW invariants obtained through other mathematical means (when possible)
- Theoretical evidence: the B-model can be understood as a quantum Kodaira-Spencer theory [BCOV], which, for Y, reduces to the quantum theory of a chiral boson living on the spectral curve. The amplitudes of this theory satisfy the recursion [Mariño, Dijkgraaf-Vafa]
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A tragedy of mathematics is a beautiful conjecture
ruined by an ugly fact.

Some consequences

The conjecture provides a new and unexpected recursion structure in GW theory, with many ramifications. Some examples:

- solves GW theory on toric CY threefolds
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  $\longrightarrow$ integrable systems? [Dubrovin, Givental]
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Simple Hurwitz numbers:

- count number of covers of $\mathbb{P}^1$ by genus g Riemann surfaces with arbitrary ramification at one point
- generating functions of simple Hurwitz numbers can be obtained as a particular limit of open A-model topological string theory on $\mathbb{C}^3$ (topological vertex) [Mariño-Vafa, Liu-Liu-Zhou]

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