Tautological Ring of Moduli Spaces of Curves

Hao Xu

Talk at Colloquium of University of California at Davis

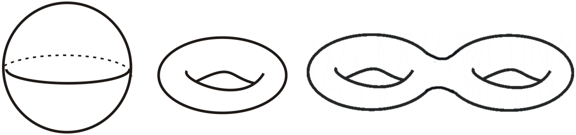
April 12, 2010, 4:10pm

Outline of the presentation

- Basics of moduli spaces of curves and its intersection theory, especially the integrals of ψ classes.
- We will present our work (with Prof. Kefeng Liu) on n-point functions, higher Weil-Petersson volumes, Faber intersection number conjecture, explicit tautological relations in the moduli spaces of curves. The starting point of our work is the Witten-Kontsevich theorem relating intersection theory and integrable systems.

Riemann surfaces

A Riemann surfaces is a 2-dimensional complex manifolds (or smooth algebraic curve over $\mathbb{C}$).



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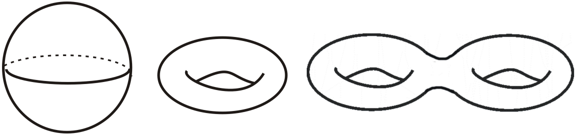

Let $n \geq 1$. The Fermat curve

$$\{(x, y, z) \in \mathbb{P}^2 \mid x^n + y^n = z^n\}.$$

is a Riemann surface with genus $(n-1)(n-2)/2$.

Stable curves

A stable curve with n marked points is a connected and compact curve with nodes with n smooth points labeled by $\{1, \dots, n\}$

$$\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$$

and satisfy:

(i) each genus 0 component has at least 3 node-branches or marked points;

(ii) each genus 1 component has at least one node-branch or marked point.

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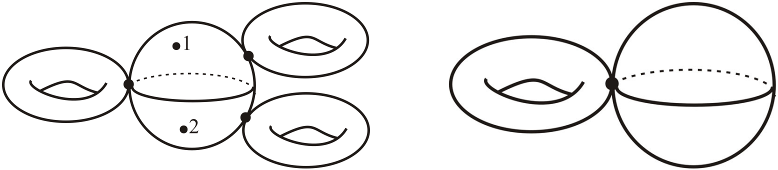

a stable curve in $\overline{\mathcal{M}}_{3,2}$ — unstable curve

Fine moduli spaces of curves does not exist

Consider contravariant functors

$$Scheme \longrightarrow Sets$$

$$\begin{aligned} Hom(\bullet, S): &\quad B \longrightarrow Hom(B,S) \ \ \text{(S is a scheme)} \\ F_V: &\quad B \longrightarrow \text{(family of k-dim subspaces of V)} \\ F_{M_g}: &\quad B \longrightarrow \ \text{(family of genus g curves over } B) \end{aligned}$$

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The grassmannian Grass(k,n) is the fine moduli space of F_V.

Fine moduli space of curves over $\mathbb{C}$ fail to exist due to automorphisms.

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Compactification of moduli spaces of curves

Constructions of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$

- The quotient of Teichmüller spaces by the action of the mapping class group.
- Geometric invariant theory (Mumford, Gieseker, etc).

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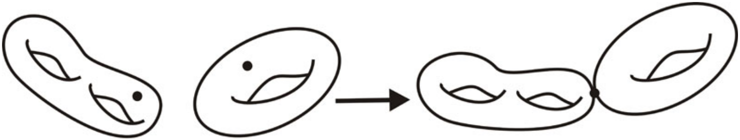
The description of $\partial\overline{\mathcal{M}}_g$ indicates that it is natural to consider curves with marked points.

Moduli spaces of curves with marked points

There are two type of boundary morphisms:

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$



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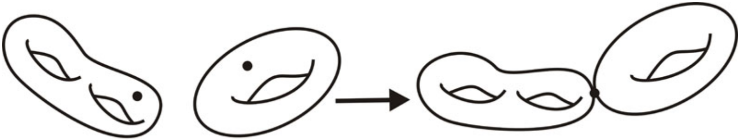
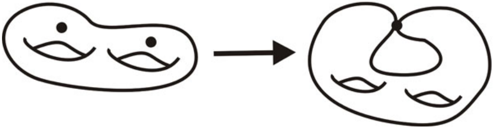
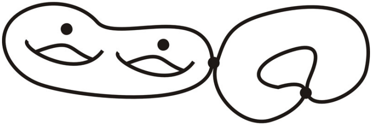


$$\overline{\mathcal{M}}_{g,n+2} \longrightarrow \overline{\mathcal{M}}_{g-1,n}$$



Dual graph of nodal curves

The following is a nodal curve of genus 3 with two components and two marked points.



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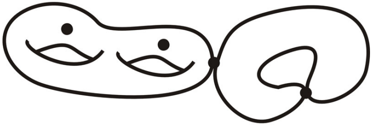

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Its associated dual graph is:

Dual graph also denotes the class of the corresponding strata in moduli space.

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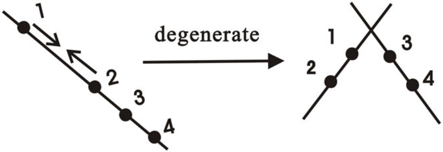

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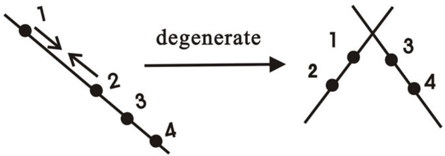
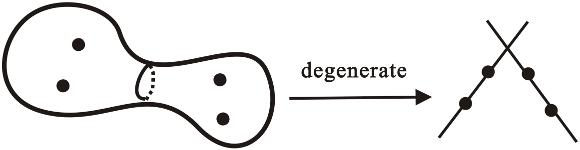
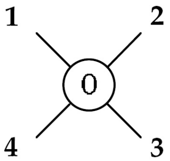
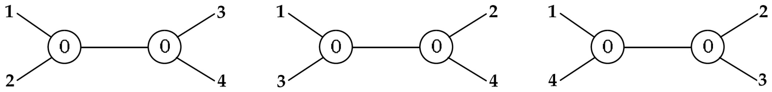




The strata of $\overline{\mathcal{M}}_{0,4}$



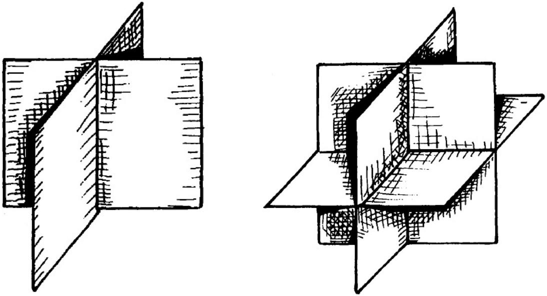
stratum of nonsingular curves



boundary strata

In fact, we have $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$.

Intersection theory



(Bézout theorem) Let Y, Z be generic distinct curves in $\mathbb{P}^2$, having degree d and e. Then the number of intersection points of Y, Z is de.

Intersection theory on moduli spaces of curves

Mumford defined the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ on moduli spaces of curves in 1983. He emphasized the tautological subring:

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}),$$

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For $\alpha, \beta \in A^*(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$, define

$$\alpha \cdot \beta = \frac{1}{\# G}(p_*[p^*\alpha \cdot p^*\beta]).$$

Tautological classes on $\overline{\mathcal{M}}_{g,n}$

- $\psi_i = c_1(L_i)$, where L_i is the line bundle whose fiber over each pointed stable curve is the cotangent line at the ith marked point.

$$\begin{array}{ccc} L_i & \supset & T_i^* C \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [C] \end{array}$$

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- $\kappa_i = \pi_*(\psi_{n+1}^{i+1})$, ($\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism).

Hodge integrals

We call the following integrals the Hodge integrals on moduli spaces of curves

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle \triangleq \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

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Based on Mumford's Chern character formula

$$\mathrm{ch}_{2m-1}(\mathbb{E}) = \frac{B_{2m}}{(2m)!} \left[\kappa_{2m-1} - \sum_{i=1}^{n} \psi_i^{2m-1} + \frac{1}{2} \sum_{\xi \in \Delta} l_{\xi_*} \left(\sum_{i=0}^{2m-2} \psi_1^i (-\psi_2)^{2m-2-i} \right) \right]$$

Faber's algorithm reduces the calculation of general Hodge integrals to those with pure ψ classes.

Intersection of pure ψ classes

The following integrals are called descendent integrals:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $d_1 + \cdots + d_n = 3g - 3 + n$.

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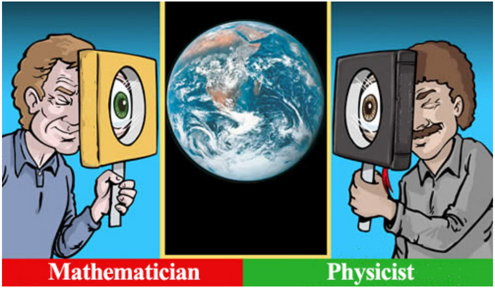

Mathematician: They are intersection numbers over varieties!

Physicist: They are correlation functions of 2-D gravity!

Numerical properties of intersection numbers

We know from Okounkov's work that

$$\sum_{g=0}^{\infty} \sum_{\sum d_j = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^{n} x_j^{d_j}$$

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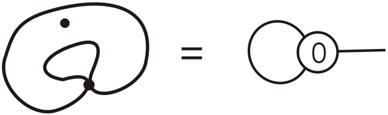
For $d_1 < d_2$, we have the multinomial value property:

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_g .$$

We have confirmed its validity for all $g \leq 20$.

Moduli of elliptic curves $\overline{\mathcal{M}}_{1,1}$

We may identify $\overline{\mathcal{M}}_{1,1}$ with the compactification of $\mathrm{PSL}(2,\mathbb{Z})\backslash H$.
The unique singular curve at ∞



The orbifold structure of $\overline{\mathcal{M}}_{1,1}$ arises from a finite quotient of the Riemann sphere $\mathbb{CP}^1$.

Compute $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1$ (Arithmetic)

The sections of Ω^n over $\overline{\mathcal{M}}_{1,1}$ correspondes exactly to entire modular forms of weight n.

The Ramanujan tau function

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad \text{where } q = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}$$

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Since generic elliptic curve has an involution,

$$deg \Omega^{12} = deg([\infty]) = \frac{1}{2}$$

hence we have

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = deg \Omega = \frac{1}{24}.$$

KdV hierarchy

The KdV hierarchy is the following hierarchy of differential equations:

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}, \qquad n \geq 1$$

where R_n are differential polynomials in $U, \dot{U}, \ddot{U}, \ldots$ defined recursively by

$$R_1 = U, \qquad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1} \left(\frac{\partial U}{\partial t_0} R_n + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} R_n \right).$$

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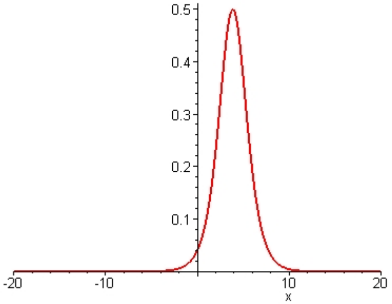
It is easy to compute R_n recursively

$$R_2 = \frac{1}{2} U^2 + \frac{1}{12} \frac{\partial^2 U}{\partial t_0^2},$$

$$R_3 = \frac{1}{6} U^3 + \frac{U}{12} \frac{\partial^3 U}{\partial t_0^3} + \frac{1}{24} (\frac{\partial U}{\partial t_0})^2 + \frac{1}{240} \frac{\partial^4 U}{\partial t_0^4},$$

$$\vdots$$

1-soliton solution of KdV hierarchy



This is the traveling wave solution

$$u(x,t) = \frac{1}{2}c \cdot \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c}(x-ct)+x_0\right)$$

of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

Scott Russell Aqueduct

On the Scott Russell Aqueduct, 12 July 1995
at Edinburgh, Scotland

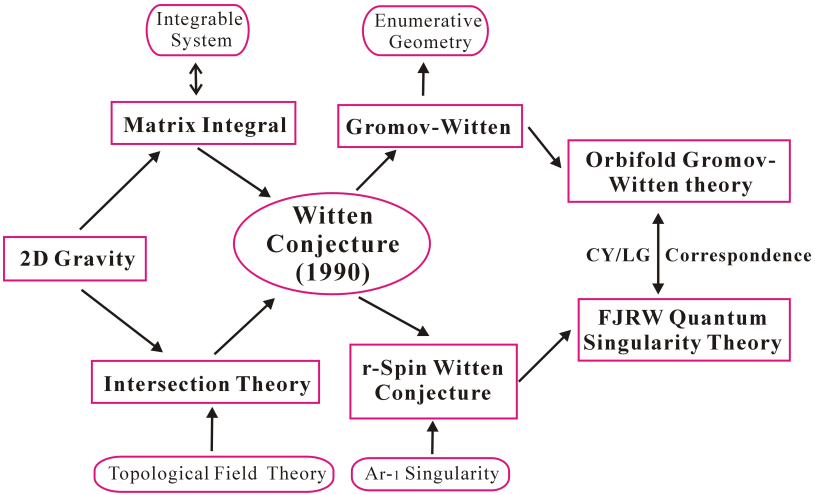
Witten's conjecture

The Witten-Kontsevich theorem states that the generating function for ψ class intersection numbers

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}=(n_0,n_1,\cdots)} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy, i.e. $U \triangleq \partial^2 F / \partial t_0^2$ obeys all equations in the KdV hierarchy.

A diagram



Integrable System
Enumerative Geometry
Matrix Integral
Gromov-Witten
Orbifold Gromov-Witten theory
2D Gravity
Witten Conjecture (1990)
CY/LG Correspondence
FJRW Quantum Singularity Theory
Intersection Theory
r-Spin Witten Conjecture
Topological Field Theory
Ar-1 Singularity

The n-point function

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Note that n-point functions encoded all information of intersection numbers on moduli spaces of curves. Note its difference with Witten's "free energy"

$$F(t_0,t_1,\dots)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\frac{t_{d_1}\cdots t_{d_n}}{n!}.$$

Recursive formula of n-point functions

Let $n \geq 2$.

$$G(x_1, \ldots, x_n) = \sum_{r,s \geq 0} \frac{(2r+n-3)!! P_r(x_1, \ldots, x_n) \Delta(x_1, \ldots, x_n)^s}{4^s (2r+2s+n-1)!!}$$

where P_r and Δ are homogeneous symmetric polynomials

$$\Delta = \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$P_r = \left(\frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I \coprod J} (\sum_{i \in I} x_i)^2 (\sum_{i \in J} x_i)^2 G(x_I) G(x_J) \right)_{3r+n-3}$$

$$= \frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I \coprod J} (\sum_{i \in I} x_i)^2 (\sum_{i \in J} x_i)^2 \sum_{r'=0}^{r} G_{r'}(x_I) G_{r-r'}(x_J).$$

New effective recursion formula

$$
\begin{aligned}
&(2g+n-1)(2g+n-2)\langle\prod_{j=1}^{n}\tau_{d_j}\rangle_g \\
&\quad=\frac{2d_1+3}{12}\langle\tau_0^4\tau_{d_1+1}\prod_{j=2}^{n}\tau_{d_j}\rangle_{g-1}-\frac{2g+n-1}{6}\langle\tau_0^3\prod_{j=1}^{n}\tau_{d_j}\rangle_{g-1}\\
&\qquad+\sum_{\{2,\ldots,n\}=I\coprod J}(2d_1+3)\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\\
&\qquad\quad-\sum_{\{2,\ldots,n\}=I\coprod J}(2g+n-1)\langle\tau_{d_1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
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When $d_j \geq 1$, all non-zero intesection numbers on RHS have genera strictly less than g.

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&\qquad + \sum_{\{2,\ldots,n\}=I\coprod J}(2d_1+3)\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\
&\qquad\quad - \sum_{\{2,\ldots,n\}=I\coprod J}(2g+n-1)\langle\tau_{d_1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

When $d_j \geq 1$, all non-zero intesection numbers on RHS have genera strictly less than g.

This formula of us is used in Stephani Yang's Macaulay2 package "KaLaPs".

Why integrals of ψ classes

Intersection theory on moduli spaces of curves has several applications and connections:

- Tautological ring of moduli spaces of curves (Faber's conjecture)
- Gromov-Witten theory
- Landau-Ginzburg theory (FJRW invariants)
- Hurwitz numbers

$$V_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{3g-3+n}.$$

Mathematician: They are integrals of Weil-Petersson metric!

Physicist: They are multiloop amplitudes of Polyakov string!

Mirzakhani's recursion formula of Weil-Petersson volumes

Mirzakhani proved a beautiful recursion formula for the Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$ of genus g hyperbolic surfaces with n geodesic boundary components of specified length $\mathbf{L} = (L_1, \ldots, L_n)$.

$$\begin{aligned}
\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I \coprod J}} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} xyH(t, x+y) \\
\times \mathrm{Vol}_{g_1,n_1}(x, \mathbf{L}_I)\mathrm{Vol}_{g_2,n_2}(y, \mathbf{L}_J)dxdydt \\
+ \frac{1}{2L_1} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} xyH(t, x+y)\mathrm{Vol}_{g-1,n+1}(x, y, L_2, \ldots, L_n)dxdydt \\
+ \frac{1}{2L_1} \sum_{j=2}^{n} \int_0^{L_1} \int_0^{\infty} x\big(H(x, L_1+L_j) + H(x, L_1-L_j)\big) \\
\times \mathrm{Vol}_{g,n-1}(x, L_2, \ldots, \hat{L}_j, \ldots, L_n)dxdt,
\end{aligned}$$

where the kernel function

$$H(x, y) = \frac{1}{1+e^{(x+y)/2}} + \frac{1}{1+e^{(x-y)/2}}.$$

Mulase-Safnuk differential form of Mirzakhani's recursion

$$
\begin{aligned}
&(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^{a}\rangle_g \\
&= \sum_{j=2}^{n}\sum_{b=0}^{a}\frac{a!}{(a-b)!}\frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!}\beta_b\langle\kappa_1^{a-b}\tau_{b+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g \\
&+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1} \\
&\quad+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b \\
&\qquad\qquad\times\langle\kappa_1^{c}\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},
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&+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1} \\
&\quad+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b \\
&\qquad\times\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},
\end{aligned}
$$

$$
\beta_b=(2^{2b+1}-4)\frac{\zeta(2b)}{(2\pi^2)^b}=(-1)^{b-1}2^b(2^{2b}-2)\frac{B_{2b}}{(2b)!}.
$$

Recursion formula of Higher Weil-Petersson Volumes

$\mathbf{m} = (m_1, m_2, \dots) \in N^\infty$, define

$$|\mathbf{m}| := \sum_{i\geq 1} i \cdot m_i, \quad ||\mathbf{m}|| := \sum_{i\geq 1} m_i$$

We proved the following recursion formula of Higher WP Volumes.

$$\begin{aligned}
&(2d_1+1)!!\langle \kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g \\
&= \sum_{j=2}^{n} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i}\rangle_g \\
&+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i=2}^{n} \tau_{d_i}\rangle_{g-1} \\
&\quad + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I\coprod J=\{2,\dots,n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}} (2r+1)!!(2s+1)!! \\
&\qquad \times \langle \kappa(\mathbf{e})\tau_r \prod_{i\in I} \tau_{d_i}\rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}$$

Tautological rings

Denote by $\mathcal{M}_g$ the moduli space of Riemann surfaces of genus $g \geq 2$. The tautological ring $R^*(\mathcal{M}_g)$ is defined to be the $\mathbb{Q}$-subalgebra of the Chow ring $\mathcal{A}^*(\mathcal{M}_g)$ generated by the tautological classes κ_i and λ_i.

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$R^*(\mathcal{M}_g)$ has the following properties:

i) (Mumford) $R^*(\mathcal{M}_g)$ is in fact generated by the $g-2$ classes $\kappa_1, \ldots, \kappa_{g-2}$;

ii) (Looijenga) $R^j(\mathcal{M}_g) = 0$ for $j > g-2$ and $\dim R^{g-2}(\mathcal{M}_g) \leq 1$ (Faber showed that actually $\dim R^{g-2}(\mathcal{M}_g) = 1$).

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$$R^*(\mathcal{M}_g) = \frac{\mathbb{C}[\kappa_1, \ldots, \kappa_{g-2}]}{\mathcal{I}},$$

where $\mathcal{I}$ is the ideal of tautological relations.

Faber's conjecture

Around 1993, Faber proposed a series of remarkable conjectures about the structure of the tautological ring $R^*(\mathcal{M}_g)$. It is a major conjecture in the subject of moduli spaces of curves.

Roughly speaking, Faber's conjecture asserts that $R^*(\mathcal{M}_g)$ behaves like the cohomology ring of a $(g-2)$-dimensional complex projective manifold.

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Roughly speaking, Faber's conjecture asserts that $R^*(\mathcal{M}_g)$ behaves like the cohomology ring of a $(g-2)$-dimensional complex projective manifold.

i) (Perfect pairing conjecture) When an isomorphism $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$ is fixed, the following natural pairing is perfect

$$R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \longrightarrow R^{g-2}(\mathcal{M}_g) = \mathbb{Q};$$

Faber's perfect paring conjecture is still open to this day.

ii) The [g/3] classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the ring, with no relations in degrees $\leq [g/3]$; (proved by Morita and Ionel)

Faber intersection number conjecture

An important part (the only quantitative part) of Faber's conjecture is the famous Faber intersection number conjecture.

$$\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!\prod_{j=1}^{n}(2d_j-1)!!} = \langle\tau_{2g}\prod_{j=1}^{n}\tau_{d_j}\rangle_g$$

$$-\sum_{j=1}^{n}\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g + \frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

$\lambda_g \lambda_{g-1}$ theorem

Faber intersection number conjecture is equivalent to

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)! \prod_{j=1}^{n}(2d_j-1)!!},$$

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This was proved by Getzler and Pandharipande (1998) conditional to Givental's work on Virasoro conjecture for $\mathbb{P}^n$. Recently Teleman announced a proof of the Virasoro conjecture for all manifolds with semi-simple quantum cohomology using the Mumford conjecture.

Tautological relations in $R^*(\mathcal{M}_g)$

Faber intersection number conjecture is equivalent to the following tautological relation in $R^*(\mathcal{M}_g)$

$$\pi_*(\psi_1^{d_1+1}\dots\psi_n^{d_n+1}) = \sum_{\sigma\in S_n}\kappa_\sigma = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^n(2d_j+1)!!}\kappa_{g-2},$$

where $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the forgetful morphism.

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where $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the forgetful morphism.

In 2006, Goulden, Jackson and Vakil give an enlightening proof of this relation for up to three points. Their remarkable proof relied on relative virtual localization in Gromov-Witten theory and some tour de force combinatorial computations.

Importance

The Faber intersection number determines the ring structure of $R^*(\mathcal{M}_g)$ if Faber's perfect pairing conjecture is true

$$R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

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is a perfect pairing.

Counterexamples of analogues of Faber's perfect pairing conjecture on partially compactified moduli spaces of curves have recently been found by R. Cavalieri and S. Yang.

Proof of Faber intersection number conjecture

Now we describe our proof.

Since one and two-point functions in genus 0 are

$$F_0(x) = \frac{1}{x^2}, \qquad F_0(x, y) = \frac{1}{x + y} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{y^{k+1}},$$

it is consistent to define the virtual intersection numbers

$$\langle \tau_{-2} \rangle_0 = 1, \qquad \langle \tau_k \tau_{-1-k} \rangle_0 = (-1)^k, \ k \geq 0.$$

Relations with n-point functions

i) $$\frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_{g-1}=[F_{g-1}(y,-y,x_1,\dots,x_n)]_{y^{2g-2}}$$

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ii) $$\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}+\langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g}\rangle_g$$

$$-\sum_{j=1}^{n}\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g$$

$$=\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j\in\mathbb{Z}}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$

$$=\left[\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}F_{g'}(y,x_I)F_{g-g'}(-y,x_J)\right]_{y^{2g-2}\prod_{i=1}^{n}x_i^{d_i}}$$

Explicit tautological relation in $R^{g-2}(\mathcal{M}_g)$

Let $g \geq 3$ and $|\mathbf{m}| = g-2$. Then the following relation

$$\kappa(\mathbf{m}) = \frac{1}{(||\mathbf{m}||-1)} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ ||\mathbf{L}||\geq 2}} A_{g,\mathbf{L}} \binom{\mathbf{m}}{\mathbf{L}} \kappa(\mathbf{L}' + \delta_{|\mathbf{L}|}), \tag{1}$$

where $A_{g,\mathbf{L}}$ are some explicitly known constants.

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where $A_{g,\mathbf{L}}$ are some explicitly known constants.

Note that in the right-hand side, $||\mathbf{L}' + \boldsymbol{\delta}_{|\mathbf{L}|}|| < ||\mathbf{m}||$, so it is indeed an effective recursion relation.

Another tautological relation in $R^*(\mathcal{M}_g)$

Let $|\mathbf{m}| \leq g-2$. Then

$$|\mathbf{m}| F_g(\mathbf{m}) = (g-1) \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \mathbf{L} \neq \mathbf{0}}} C_{\mathbf{L}} F_g(\mathbf{L}'),$$

where $F_g(\mathbf{0}) = 1$ and $C_{\mathbf{L}}$ are some explicitly known constants.

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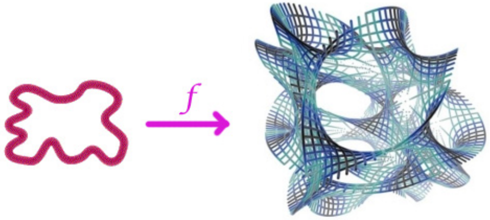
where $F_g(\mathbf{0}) = 1$ and $C_{\mathbf{L}}$ are some explicitly known constants.

So these virtual numbers $F_g(\mathbf{m})$ when $|\mathbf{m}| < g-2$ are useful.

Gromov-Witten invariants

If $\gamma_{a_1},\dots,\gamma_{a_n}\in H^*(X,\mathbb{Q})$, the Gromov-Witten invariants are defined by

$$\langle\tau_{d_1}(\gamma_{a_1})\dots\tau_{d_n}(\gamma_{a_n})\rangle_{g,\beta}^X=\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}}\Psi_1^{d_1}\cdots\Psi_n^{d_n}\cup\mathrm{ev}^*(\gamma_{a_1}\boxtimes\cdots\boxtimes\gamma_{a_n}).$$



Mathematician: They are virtual enumerative numbers of curves!

Physicist: They are path integrals of the topological field theory!

Universal relations of Gromov-Witten invariants

String equation

$$\langle \tau_{0,0}\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X = \sum_{i=1}^n \langle \tau_{k_1,a_1}\cdots\tau_{k_i-1,a_i}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X.$$

Dilaton equation

$$\langle \tau_{1,0}\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X = (2g-2+n)\langle \tau_{k_1,\alpha_1}\cdots\tau_{k_n,\alpha_n}\rangle_{g,\beta}^X.$$

Divisor equation

$$\begin{aligned}\langle \tau_0(\omega)\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X &= (\omega\cap\beta)\langle \tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X \\ &\quad + \sum_{i=1}^n \langle \tau_{k_1,a_1}\cdots\tau_{k_i-1}(\omega\cup\gamma_a)\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X,\end{aligned}$$

where $\omega \in H^2(X,\mathbb{Q})$.

Topological recursion relations

We may pull back tautological relations on $\overline{\mathcal{M}}_{g,n}$ via the forgetful map

$$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$$

to get universal equations for Gromov-Witten invariants by the splitting axiom and cotangent line comparison equations.

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We may pull back tautological relations on $\overline{\mathcal{M}}_{g,n}$ via the forgetful map

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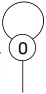
to get universal equations for Gromov-Witten invariants by the splitting axiom and cotangent line comparison equations.

On $\overline{\mathcal{M}}_{1,1}$, the relation

$$\psi_1 = \frac{1}{12}$$ 

implies the genus 1 TRR

$$\langle\langle \tau_k(x) \rangle\rangle_1 = \langle\langle \tau_{k-1}(x)\gamma_\alpha \rangle\rangle_0 \langle\langle \gamma^\alpha \rangle\rangle_1 + \frac{1}{24} \langle\langle \tau_{k-1}(x)\gamma_\alpha\gamma^\alpha \rangle\rangle_0.$$

WDVV equation

From the simple fact that the three boundary divisors of $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ are equal, there is the well-known

WDVV equation

$$\langle\langle \tau_{k_1,a_1}\tau_{k_2,a_2}\gamma_\alpha\rangle\rangle_0 \langle\langle \gamma^\alpha \tau_{k_3,a_3}\tau_{k_4,a_4}\rangle\rangle_0 = \langle\langle \tau_{k_1,a_1}\tau_{k_3,a_3}\gamma_\alpha\rangle\rangle_0 \langle\langle \gamma^\alpha \tau_{k_2,a_2}\tau_{k_4,a_4}\rangle\rangle_0$$

which is the associativity condition of the quantum cohomology ring.

Vanishing Identities of Gromov-Witten Invariants

Our proof of Faber intersection number conjecture ($X = pt$) motivates us to formulate the following conjecture

Conjecture

Let $x_i, y_i \in H^*(X)$ and $k \geq 2g-3+r+s$. Then

$$\sum_{g'=0}^{g} \sum_{j\in\mathbb{Z}} (-1)^j \langle\langle \tau_j(\gamma_a) \prod_{i=1}^{r} \tau_{p_i}(x_i) \rangle\rangle_{g'} \langle\langle \tau_{k-j}(\gamma^a) \prod_{i=1}^{s} \tau_{q_i}(y_i) \rangle\rangle_{g-g'} = 0.$$

Note that j runs over all integers.

where we adopt Gathmann's convention $\langle \tau_{-2}(pt) \rangle_{0,0}^{X} = 1$ and

$$\langle \tau_m(\gamma_1) \tau_{-1-m}(\gamma_2) \rangle_{0,0}^{X} = (-1)^{\max(m,-1-m)} \int_X \gamma_1 \cdot \gamma_2, \quad m \in \mathbb{Z}.$$

The conjecture has recently been proved recently by Xiaobo Liu and R. Pandharipande.

Virasoro conjecture

Let $r = \dim X$ and $\{\gamma_a\}$ a basis in $H^*(X)$.

i) $R_a^b \gamma_b = c_1(X) \cup \gamma_a$, $\tilde{t}_k^a = t_k^a - \delta_{a0}\delta_{k1}$

ii) $[x]_i^k = e_{k+1-i}(x, x+1, \ldots, x+k)$

iii) If $\gamma_a \in H^{p_a,q_a}(X)$, $b_a = p_a + (1-r)/2$.

$$\begin{aligned} L_k = \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} \Big(& [b_a + m]_i^k (R^i)_a^b \tilde{t}_m^a \partial_{b,m+k-i} \\ & + \frac{\hbar}{2} (-1)^{m+1} [b_a - m - 1]_i^k (R^i)^{ab} \partial_{a,m} \partial_{b,k-m-i-1} \Big) \\ + \frac{1}{2\hbar} (R^{k+1})_{ab} t_0^a t_0^b & + \frac{\delta_{k0}}{48} \int_X ((3-r) c_r(X) - 2 c_1(X) c_{r-1}(X)). \end{aligned}$$

Virasoro conjecture (Eguchi, Hori and Xiong and also by Katz)

$$L_k(\exp F) = 0, \quad k \geq -1.$$

Eynard-Orantin's recursion

- Originated from random matrix theory, Eynard and Orantin developed a theory of symplectic invariants of curves.
- Bouchard-Klemm-Mariño-Pasquetti (BKMP) further conjectures that Eynard-Orantin's recursion can be used to compute the Gromov-Witten invariants of toric Calabi-Yau 3-folds through mirror symmetry.
- Bouchard-Mariño conjectured a recursive formula for simple Hurwitz numbers based on BKMP.
- Bouchard-Mariño conjecture has been proved recently by Borot, Eynard, Mulase and Safnuk.

ADE singularities

$$A_n : W = x^{n+1}, \ \ n \geq 1$$
$$D_{n+2} : W = x^{n+1} + xy^2, \ \ n \geq 2$$
$$E_6 : W = x^3 + y^4$$
$$E_7 : W = x^3 + xy^3$$
$$E_8 : W = x^3 + y^5$$

Mathematician: They are ADE singularities!

Physicist: They are potential functions of Landau-Ginzburg theory!

Landau-Ginzburg model

In particle physics, any quantum field theory with a unique classical vacuum state and a potential energy with a degenerate critical point is called a Landau-Ginzburg theory.

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- (Greene-Vafa-Warner) Landau-Ginzburg theory is related by a renormalization group flow to sigma models on Calabi-Yau manifolds.
- (Witten) Landau-Ginzburg theory and sigma model on Calabi-Yau manifolds are different phases of the same theory, similar to the relation between the gas and the liquid phases of a fluid.

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Examples include:

- The quintic Calabi-Yau 3-fold $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$
- Orbifold Calabi-Yau in weighted projective spaces.
- Simple, unimodal and bimodal singularities.

Landau-Ginzburg model

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$$W : \mathbb{C}^N \to \mathbb{C}$$

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$$W : \mathbb{C}^N \to \mathbb{C}$$

with unique weights and an isolated singularity at the origin.

There are weights (or charges) $q_1, \ldots, q_N$ such that

$$W(\lambda^{q_1} x_1, \ldots, \lambda^{q_N} x_N) = \lambda W(x_1, \ldots, x_N)$$

for all $\lambda \in \mathbb{C}$ with central charge $\hat{c}_W = \sum(1 - 2q_i)$.

Landau-Ginzburg B-model

The Landau-Ginzburg B-model is the Milnor ring (or Chiral ring):

$$\mathscr{Q}_W := \frac{\mathbb{C}[x_1, \ldots, x_N]}{\left(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N}\right)}$$

A local Artin ring.

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A local Artin ring.

The unique highest degree element is

$$\det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)$$

Landau-Ginzburg A-model

For any quasi-homogeneous, non-degenerate singularity W and an admissible group G of diagonal symmetry of W. Following a suggestion of Witten, Fan, Jarvis and Ruan constructed

i) FJRW state space (a Frobenius algebra)

$$\mathscr{H}_{W,G} = \oplus_{g\in G}\mathscr{H}_g, \quad \text{where } \mathscr{H}_g = H^{mid}(\mathrm{Fix}g, W_g^{\infty}, \mathbb{C})^G$$

ii) Moduli of stable W-orbicurves and its virual cycles.

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ii) Moduli of stable W-orbicurves and its virual cycles.

The associated cohomological field theory defines Gromov-Witten type invariants for W.

Intersection theory and singularities

Table: Simple and elliptic singularities

W	$\hat{c}_W$	moduli of W curves	integrable hierarchy
A_1	0	$\overline{\mathcal{M}}_{g,n}$	KdV
$A_{r-1},\ r \geq 2$	$\frac{r-2}{r}$	$\overline{\mathcal{M}}_{g,n}^{1/r}$	Gelfand-Dickey
$D_n,\ n \geq 4$	$\frac{n-2}{n-1}$		Drinfeld-Sokolov Kac-Wakimoto
E_6	$\frac{5}{6}$		
E_7	$\frac{8}{9}$		
E_8	$\frac{14}{15}$		
$P_8 = x^3 + y^3 + z^3$	1		Don't know yet
$X_9 = x^4 + y^4$	1		
$J_{10} = x^3 + y^6$	1		

Thank you!