AN INVITATION TO 2D TQFT AND QUANTIZATION OF HITCHIN SPECTRAL CURVES

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ABSTRACT. This article consists of two parts. In Part 1, we present a formulation of two-dimensional topological quantum field theories in terms of a functor from a category of ribbon graphs to the endofuntor category of a monoidal category. The key point is that the category of ribbon graphs produces all Frobenius objects. Necessary backgrounds from Frobenius algebras, topological quantum field theories, and cohomological field theories are reviewed. A result on Frobenius algebra twisted topological recursion is included at the end of Part 1.

In Part 2, we explain a geometric theory of quantum curves. The focus is placed on the process of quantization as a passage from families of Hitchin spectral curves to families of opers. To make the presentation simpler, we unfold the story using $SL_2(\mathbb{C})$-opers and rank 2 Higgs bundles defined on a compact Riemann surface C of genus greater than 1. In this case, quantum curves, opers, and projective structures in C all become the same notion. Background materials on projective coordinate systems, Higgs bundles, opers, and non-Abelian Hodge correspondence are explained.

CONTENTS

0. Preface: An inspiration from the past

A recent discovery of a cuneiform tablet dated around 350 to 50 B.C.E. suggests that ancient Babylonians must have used geometry of *time-momentum space* to establish accurate calculations of Jupiter's orbit [94]. This impressive paper also contains the picture of the tablet which describes the Babylonian's method of *integration*.
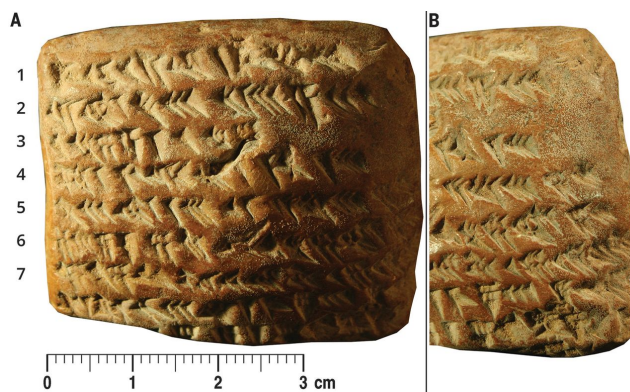

FIGURE 0.1. The cuneiform tablet used in the analysis of [94].

The orbit of Jupiter is a graph in coordinate space-time. By considering the graph of momentum of Jupiter in time-momentum space, Babylonians *visualized* the integral of the momentum by the area underneath the curve, and using a trapezoidal approximation, they actually obtained an estimated value of the integral. This gives a prototype of Newton's Fundamental Theorem of Calculus. This idea of Babylonians relating geometry of time-momentum space with the analysis of actual orbit of Jupiter is striking, because it suggests their equal treatment of coordinate space and momentum space. Although it is a stretch, we could imagine the very foundation of *symplectic geometry* here.

Many mathematical cuneiform tablets recording numbers and algebraic calculations have been our source of imagination. The most famous is **Plimpton 322** of around 1,800 B.C.E. (see Figure 0.2). It lists 15 Pythagorean numbers in increasing order of hypotenuse angles from about 45 degrees to 60 degrees [97].

FIGURE 0.2. Plimpton 322

Babylonians seem to have known an algorithm to calculate an approximate value of the square root of any number. For example, there is a cuneiform tablet that shows the sexagesimal expansion of $\sqrt{2}$. Although there have been many speculations for practical purposes of Plimpton 322 and mechanisms to come up with the listed numbers, our imagination goes to the surprise of the creator of the tablet. For the pair of numbers (y, z) listed in the second and the third columns, $z^2 - y^2$ is always a perfect square. Therefore, the square root algorithm terminates in a finite number of steps for these values, and gives an exact answer. They must have found *finiteness in the forest of infinity*. This is a strong sentiment that resonates with our mind today.

The tablet of Figure 0.1 does not show any geometry. The author of [94] convincingly argues that behind the list of these seemingly meaningless numbers on the tablets, there is a profound geometric investigation of the planetary motion. Our imagination is piqued by the discovery: the interplay between algebra, geometry, and astronomy; and the equal treatment of momentum space and coordinate space. What the feeling of the authors of the tablet would have been, when they were creating it? It must have been a sharp happiness of discovery that mathematics correctly predicts the very nature surrounding us, such as planetary motion.

The accuracy of calculations of Babylonians is another astonishment, in addition to the lack of pictures. It is in sharp contrast to Euclid's **Elements**, which was written around the same time in Greece. In *Elements*, we see many beautiful geometric pictures, and *proofs*. The discovery of *axiomatic method* culminated in it. This dichotomy, being able to **calculate** a quantity to a high precision, vs. having a **proof** of the formula based on a **finiteness property**, is our very motivation for writing these notes. There are a lot of amazing formulas around us in the direction that we describe. At this moment, we do not have the final understanding of them, yet.

In the first part of these lectures, we explore two-dimensional topological quantum field theories formulated in terms of *cell graphs*. A cell graph of type (g, n) is the 1-skeleton of a cell-decomposition of a compact oriented topological surface of genus $g \geq 0$ with n labeled 0-cells, or vertices. Such graphs are called by many different names: ribbon graphs, dessins d'enfants, maps, embedded graphs, etc. We use the terminology "cell graph" indicating the different nature of the graphs on surfaces, which record degeneration of surfaces. Although a cell graph is a 1-dimensional object, the notion of a *face* is well defined as a 2-cell of the cell decomposition that the cell graph defines on a given surface.

The degree of a vertex of a cell graph is the number of incident half-edges. We call a 1-cell an edge of the graph. As explained in our previous lecture notes [31], the number of cell graphs of type $(0, 1)$ with the unique vertex of degree $2m$ is $\frac{1}{2m}C_m$, where

$$C_m = \frac{1}{m+1}\binom{2m}{m}$$

is the m-th *Catalan number*. A generating function

$$z = z(x) = \sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}}$$

of Catalan numbers satisfies an algebraic equation

(0.1) $$x = z + \frac{1}{z}.$$

The story of [31, Section 2] tells us that the enumeration problem of cell graphs of arbitrary type (g, n) is solved by the **quantization** of the algebraic curve (0.1). The key formula for

the quantization comes from the Laplace transform of a combinatorial equation [31, Catalan Recursion, Section 2.1]. There, the combinatorial formula is derived by analyzing the effect of **edge-contraction operations** on cell graphs.

The new story we wish to tell in these lectures is that the *category* of cell graphs carries the information of *all* two-dimensional topological quantum field theories. In Part 1 of this article, we will present an axiomatic setup of 2D TQFT. The key idea is simple: exact same edge-contraction operations characterize Frobenius algebras. When we contract an edge of a cell graph, two vertices collide. It represents *multiplication*. When we shrink a loop, since it is a cycle on a topological surface, the process breaks a vertex into two vertices. This is the operation of *comultiplication*. The topological structure of a cell graph makes these operations dual to one another in the context of Frobenius algebras.

We start with defining Frobenius algebras. Topological quantum field theories (TQFT) and cohomological field theories (CohFT) are introduced in the following sections. A category of cell graphs is defined. We explain that this category generates all Frobenius algebras. We then make a connection between cell graphs and 2D TQFT. The theory of *topological recursion* is a leitmotif in this article. It is another manifestation of a quantization procedure. Since there are many review articles on topological recursion, we touch the subject only tangentially in this article. Our new result related to this topic is the 2D TQFT-twisted topological recursion, which is explained in the end of Part 1.

The cohomology ring of a closed oriented manifold X is a Frobenius algebra. When we consider an even dimensional manifold and only even degree cohomologies

$$(0.2) \qquad\qquad A = H^{even}(X, \mathbb{Q}),$$

we have a commutative Frobenius algebra, which is equivalent to a 2D TQFT. The *Gromov-Witten invariants* of X of genus 0 determine a quantum deformation of the ring A, known as the *big quantum cohomology*. This quantum structure then defines a holomorphic object Y, the *mirror* of X. Thus the holomorphic geometry of Y captures quantum cohomology of X. Gromov-Witten invariants are generalized to arbitrary genera. Here arises a question:

Question 0.1. *What should be the holomorphic geometry on Y that corresponds to higher-genus Gromov-Witten theory of X?*

If we consider the passage from genus 0 to higher genera Gromov-Witten theory as *quantization*, then on the side on Y, we need a quantized holomorphic geometry. A good candidate is $\mathcal{D}$-module theory. The simplest such theory fitting to this context is the notion of **quantum curves**. The relation between quantum curves and B-model geometry is considered in [2, 20, 18, 28, 29, 31, 50, 52] and others. The key point for this idea to work is when the holomorphic geometry Y corresponding to the Gromov-Witten theory of X is captured by an algebraic or analytic *curve*. These curves often appear in other areas of mathematics as *spectral curves*, such as integrable systems, random matrix theory, and Hitchin's theory of Higgs bundles. A spectral curve naturally comes with a projection to a *base curve*, making it a covering of the base curve. A quantum curve is a $\mathcal{D}$-module on this base curve, quantizing the spectral curve.

Since a CohFT based on the Frobenius algebra $H^{even}(X, \mathbb{Q})$ plays a role similar to Gromov-Witten theory of X, CohFT is a quantization of 2D TQFT. This process consists of two steps of quantizations. The first one is from the classical cohomology ring A to its quantum deformation, which is the mirror geometry. If the mirror is a curve, then the second step should be parallel to the quantization of a spectral curve to a quantum curve. We may be able to understand the reconstruction [37] of CohFT based on a semi-simple

Frobenius algebra from the 2D TQFT, that is obtained as the degree 0 part of the cohomology $H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ of the moduli space of stable curves, analogous to the construction of quantum curves.

Part 2 deals with quantum curves. Since the authors have produced a long article [31] explaining the relation between quantum curves and topological recursion, the present article is focused on geometry of the process of quantization of Hitchin spectral curves from a perspective of **opers**. Instead of providing a general theory of [26, 27, 32], we use $SL_2(\mathbb{C})$ to explain our ideas here. We will show that the classical notion of projective structures on a compact Riemann surface C of genus $g \geq 2$ studied by Gunning [53], $SL_2(\mathbb{C})$-opers, and quantum curves of Hitchin spectral curves, are indeed the exact same notion. A conjecture of Gaiotto [43] relating non-Abelian Hodge correspondence and opers, together with its solution by [27], will be briefly explained in the final section.

Part 1. **Topological Quantum Field Theory**

In Part 1 of this article, we present our new formulation of two-dimensional topological quantum field theory as a monoidal functor from a category of ribbon graphs into the endofunctor category of a monoidal category [30, 34]. We start with a quick review of a few necessary accounts from Frobenius algebras, topological quantum field theory, Gromov-Witten invariants, and cohomological field theory. There are numerous excellent textbooks available on these topics. We refer to [15, 65, 72] for more detail on these topics. We also refer to [98, 103] for the origin of TQFT in physics.

1. Frobenius algebras

Throughout this article, we denote by K a field of characteristic 0. Most of the cases we consider $K = \mathbb{Q}$ or $K = \mathbb{C}$. Let A be a finite-dimensional, unital, and associative algebra defined over K. A *bialgebra* comes with an extra set of structures, including a comultiplication. Examples of bialgebras include the group algebra $A = K[G]$ of a finite

group G. Its algebra structure is canonically determined by the multiplication rule of G. However, the group algebra has *two distinct* co-algebra structures. One of which makes $K[G]$ a Frobenius algebra, and the other makes it a Hopf algebra. This difference is reflected in which topological invariants we are dealing with. The Frobenius structure is useful for two-dimensional topology, and the Hopf structure is natural for three-dimensional topological invariants.

Let us start with defining Frobenius algebras.

Definition 1.1 (Frobenius algebra). A finite-dimensional, unital, and associative algebra A over a field K is a **Frobenius algebra** if there exists a linear map $\epsilon : A \longrightarrow K$ called **counit** such that for every $u \in A$, $\epsilon(ux) = 0$, or $\epsilon(xu) = 0$, for all $x \in A$ implies that $u = 0$. An algebra homomorphism $h : A_1 \longrightarrow A_2$ from a Frobenius algebra (A_1, ϵ_1) to another Frobenius algebra (A_2, ϵ_2) is a Frobenius homomorphism if h satisfies

(1.1)
$$
\begin{array}{ccc}
A_1 & \xrightarrow{\ h\ } & A_2 \\
{\scriptstyle \epsilon_1}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_2} \\
K & =\!=\!= & K
\end{array}.
$$

Proposition 1.2. *If A_1 and A_2 are Frobenius algebras, then $A_1 \oplus A_2$ and $A_1 \otimes A_2$ are also Frobenius algebras. Let us denote by $\mathcal{A}$ the category of Frobenius algebras over K, with morphisms defined by (1.1). Then $\mathcal{C} = (\mathcal{A}, \otimes, K)$ is a monoidal category.*

Proof. Denote by $\epsilon_i : A_i \longrightarrow K$ the counit of A_i, $i = 1, 2$. Then $(A_1 \oplus A_2, \epsilon_1 + \epsilon_2)$ is a Frobenius algebra. Similarly,

$$
\epsilon_1 \cdot \epsilon_2 : A_1 \otimes A_2 \ni u_1 \otimes u_2 \longmapsto \epsilon_1(u_1)\epsilon_2(u_2) \in K
$$

makes $A_1 \otimes A_2$ a Frobenius algebra. $\qquad\qquad\square$

Example 1.3. The one-dimensional vector space $A = K$, with the identity map $\epsilon(u) = u$, is a simple commutative Frobenius algebra. More generally, by taking the direct sum $A = K^{\oplus n}$ and defining $\epsilon(u_1, \ldots, u_n) = u_1 + \cdots + u_n \in K$, we construct a *semi-simple* commutative Frobenius algebra $K^{\oplus n}$.

Example 1.4. The full matrix algebra $A = \mathrm{Mat}_n(K)$ consisting of $n \times n$ matrices with $\epsilon(u) = \mathrm{trace}(u)$ is a non-commutative simple Frobenius algebra for $n \geq 2$.

Example 1.5. As mentioned above, the group algebra $A = K[G]$ of a finite group G is a Frobenius algebra. Here, we define $\epsilon : K[G] \longrightarrow K$ by linearly extending the map whose value for every $g \in G$ is given by

$$
\epsilon(g) = \begin{cases} 1 & g = 1 \\ 0 & g \neq 1. \end{cases}
$$

Remark 1.6. If we define $\epsilon(g) = 1$ for all $g \in G$, then the group algebra becomes a **Hopf algebra**.

Example 1.7. The cohomology ring $H^*(M, \mathbb{R})$ of an oriented compact differential manifold M of $\dim_{\mathbb{R}} M = n$ with the cup product and

$$
\epsilon : H^*(M, \mathbb{R}) \longrightarrow H^n(M, \mathbb{R}) = \mathbb{R}
$$

is a Frobenius algebra.

The above example makes us wonder if an analogue of *Poincaré pairing* exists in a general Frobenius algebra. Indeed, the counterpart is a **Frobenius bilinear form** defined by

$$(1.2) \qquad \eta : A \otimes A \longrightarrow K, \qquad \eta(u,v) = \epsilon(uv)$$

in terms of the counit $\epsilon : A \longrightarrow K$. The Frobenius form satisfies the Frobenius associativity

$$(1.3) \qquad \eta(uv,w) = \eta(u,vw), \qquad u,v,w \in A.$$

The condition for the counit ϵ of Definition 1.1 exactly means that the Frobenius form η is non-degenerate. It therefore defines a canonical isomorphism

$$(1.4) \qquad \lambda : A \xrightarrow{\sim} A^*, \qquad \langle \lambda(u), v \rangle = \eta(u,v), \qquad u,v \in A.$$

This isomorphism determines a unique **comultiplication** $\delta : A \longrightarrow A \otimes A$ by the following commutative diagram:

$$(1.5)$$

$$
\begin{array}{ccc}
A & \xrightarrow{\ \delta\ } & A \otimes A \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda \otimes \lambda} \\
A^* & \xrightarrow[\ m^*\ ]{} & A^* \otimes A^*
\end{array} \quad .
$$

Here, $m^* : A^* \longrightarrow A^* \otimes A^*$ is the dual of the multiplication operation $m : A \otimes A \longrightarrow A$ in A. Since we are not assuming the multiplication to be commutative, we define the natural pairing

$$(A^* \otimes A^*) \otimes (A \otimes A) \longrightarrow K$$

by observing the order of entities as written. For example, for $\alpha, \beta \in A^*$ and $u, v \in A$, we calculate

$$(1.6) \qquad \langle \alpha \otimes \beta, u \otimes v \rangle = \langle \alpha, \langle \beta, u \rangle v \rangle = \langle \beta, u \rangle \langle \alpha, v \rangle.$$

As the dual of the associative multiplication, the comultiplication δ is *coassociative*, i.e., it satisfies

$$(1.7)$$

$$
\begin{array}{ccc}
 & A \otimes A & \\
{\scriptstyle \delta}\nearrow & & \searrow{\scriptstyle \delta \otimes 1} \\
A & & A \otimes A \otimes A. \\
{\scriptstyle \delta}\searrow & & \nearrow{\scriptstyle 1 \otimes \delta} \\
 & A \otimes A &
\end{array}
$$

Note that the identity element $\mathbf{1} \in A$ corresponds to $\epsilon \in A^*$ by λ, i.e., $\lambda(\mathbf{1}) = \epsilon$. This is because

$$\eta(\mathbf{1}, u) = \eta(u, \mathbf{1}) = \epsilon(u).$$

We now have the full set of data $(A, \mathbf{1}, m, \epsilon, \delta)$ that defines a bialgebra. The algebra and coalgebra structures satisfy a compatibility condition, as described below.

Proposition 1.8. *The following diagram commutes:*

(1.8)
$$
\begin{array}{ccccc}
 & & A \otimes A \otimes A & & \\
 & \nearrow^{\!\!1\otimes\delta} & & \searrow^{\!\!m\otimes 1} & \\
A \otimes A & \xrightarrow{\;\;m\;\;} & A & \xrightarrow{\;\;\delta\;\;} & A \otimes A. \\
 & \searrow_{\!\!\delta\otimes 1} & & \nearrow_{\!\!1\otimes m} & \\
 & & A \otimes A \otimes A & &
\end{array}
$$

Remark 1.9. Alternatively, we can define a Frobenius algebra as a bialgebra satisfying (1.8), together with a non-degenerate Frobenius form satisfying (1.3).

Proof. Let $\langle e_1, e_2, \ldots, e_r \rangle$ be a K-basis for A, where $r = \dim_K A$. The bilinear form η defines $r \times r$ matrices

(1.9)
$$
\eta = [\eta_{ij}], \qquad \eta^{-1} = [\eta^{ij}], \qquad \eta_{ij} := \eta(e_i, e_j).
$$

It gives the canonical basis expansion of $v \in A$:

(1.10)
$$
v = \sum_{a,b} \eta(v, e_a) \eta^{ab} e_b = \sum_{a,b} \eta(e_a, v) \eta^{ba} e_b.
$$

From (1.5), we calculate

$$
\begin{aligned}
\delta(v) &= \sum_{i,j,a,b} \eta(v, e_i e_j)(\eta^{jb} e_b) \otimes (\eta^{ia} e_a) \\
&= \sum_{i,j,a,b} \eta(v e_i, e_j)(\eta^{jb} e_b) \otimes (\eta^{ia} e_a) \\
&= \sum_{i,a} (v e_i) \otimes (\eta^{ia} e_a) \\
&= (m \otimes 1)(v \otimes \delta(\mathbf{1})),
\end{aligned}
$$

using the pairing convention (1.6) and

(1.11)
$$
\delta(\mathbf{1}) = \sum_{a,b} \eta^{ab} e_a \otimes e_b.
$$

We then have

$$
\begin{aligned}
(1 \otimes m) \circ (\delta \otimes 1)(u \otimes v) &= (1 \otimes m) \left(\sum_{i,j,a,b} \eta(u, e_i e_j)(\eta^{jb} e_b) \otimes (\eta^{ia} e_a) \otimes v \right) \\
&= \sum_{i,j,a,b} \eta(u e_i, e_j)(\eta^{jb} e_b) \otimes (\eta^{ia} e_a v) \\
&= \sum_{i,j,a,b,c,d} \eta(u e_i, e_j)(\eta^{jb} e_b) \otimes \eta^{ia} \eta(e_a v, e_c) \eta^{cd} e_d \\
&= \sum_{i,a,c,d} u e_i \otimes \eta^{ia} \eta(e_a, v e_c) \eta^{cd} e_d \\
&= \sum_{c,d} (u v e_c) \otimes \eta^{cd} e_d \\
&= (m \otimes 1)(u v \otimes \delta(\mathbf{1})) = \delta(uv).
\end{aligned}
$$

Similarly,

$$(m \otimes 1) \circ (1 \otimes \delta)(u \otimes v) = (m \otimes 1) \left(u \otimes \sum_{i,j,a,b} \eta(v, e_i e_j)(\eta^{jb} e_b) \otimes (\eta^{ia} e_a) \right)$$

$$= \sum_{i,j,a,b} u\eta(ve_i, e_j)(\eta^{jb} e_b) \otimes (\eta^{ia} e_a)$$

$$= \sum_{i,a} uve_i \otimes \eta^{ia} e_a$$

$$= (m \otimes 1)(uv \otimes \delta(\mathbf{1})) = \delta(uv).$$

This completes the proof of Proposition 1.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 1.10. We note that if A is commutative, then the quantities

(1.12) $$\eta\big(e_{i_1} \cdots e_{i_j}, e_{i_{j+1}} \cdots e_n\big) = \epsilon(e_{i_1} \cdots e_{i_n}), \qquad 1 \le j < n,$$

are completely symmetric with respect to permutations of indices.

Definition 1.11 (Euler element)**.** The **Euler element** of a Frobenius algebra A is defined by

(1.13) $$\mathbf{e} := m \circ \delta(\mathbf{1}).$$

In terms of basis, it is given by

(1.14) $$\mathbf{e} = \sum_{a,b} \eta^{ab} e_a e_b.$$

The Euler element provides the genus expansion of 2D TQFT, allowing us to calculate higher genus correlation functions from the genus 0 part of the theory.

Another application of (1.10) is the following formula that relates the multiplication and comultiplication:

(1.15) $$(\lambda(u) \otimes \mathbf{1}) \, \delta(v) = uv.$$

This is because

$$(\lambda(u) \otimes \mathbf{1}) \, \delta(v) = (\lambda(u) \otimes \mathbf{1}) \sum_{a,b} (v\eta^{ab} e_a) \otimes e_b$$

$$= \sum_{a,b} \eta(u, ve_a)\eta^{ab} e_b$$

$$= \sum_{a,b} \eta(uv, e_a)\eta^{ab} e_b = uv.$$

2. TQFT

In this section, we briefly review TQFT. For the origin of TQFT in physics, we refer to [98, 103]. Although there have been explosive mathematical developments in higher dimensional topological quantum field theories mixing different dimensions during the last decade (see for example, [13, 71] and more recent work inspired by them), we restrict our attention to the two-dimensional speciality in these lectures. From now on, Frobenius algebras we consider are finite-dimensional, unital, associative, and *commutative*. It has been established (see [1, 17]) that 2D TQFT's are classified by these types of Frobenius algebras.

The axiomatic formulation of conformal and topological quantum field theories was discovered in the 1980s by Atiyah [4] and Segal [99] (see also [65, 72, 98, 101]). A $(d-1)$-dimensional TQFT is a symmetric monoidal functor Z from the monoidal category of $(d-1)$-dimensional closed (i.e., compact without boundary) oriented smooth manifolds with oriented d-dimensional cobordism as morphisms, to the monoidal category of finite-dimensional vector spaces defined over a field K. The monoidal structure in the category of $(d-1)$-dimensional smooth manifolds is defined by the operation of taking disjoint union, which is a symmetric operation. Disjoint union with the empty set is the identity of this operation. Therefore, we define the monoidal category of K-vector spaces by *symmetric tensor products*, with the field K serving as the identity operation of tensor product. The functor Z satisfies the non-triviality condition

$$Z(\emptyset) = K,$$

and maps a disjoint union of smooth $(d-1)$-dimensional manifolds to a symmetric tensor product of vector spaces.

Let us denote by M an oriented closed smooth manifold of dimension $d-1$, and by M^{op} the same manifold with the opposite orientation. The functor Z satisfies the duality condition

$$Z(M^{op}) = Z(M)^*,$$

where $Z(M)^*$ is the dual vector space of $Z(M)$. Suppose we have an oriented bordered d-dimensional smooth manifold N. The boundary ∂N is a smooth manifold of dimension $d-1$, and the complement $N \setminus \partial N$ is an oriented smooth manifold of dimension d. We give the orientation induced from N to its boundary ∂N. The TQFT functor Z then gives an element

$$Z(N) \in Z(\partial N).$$

If N is closed, i.e., $\partial N = \emptyset$, then

$$Z(N) \in Z(\emptyset) = K$$

is a number that represents a topological invariant of N.

Now consider a bordered oriented smooth manifold N_1 with boundary

$$\partial N_1 = M_1^{op} \sqcup M_2,$$

meaning that the two separate boundaries carry different orientations. We choose that M_2 is given the induced orientation from N_1, and M_1 the opposite orientation. We interpret the situation as N_1 giving an oriented cobordism from M_1 to M_2. In this case, the functor Z defines an element $Z(N_1) \in Z(M_1)^* \otimes Z(M_2)$, or equivalently, a linear map

$$Z(N_1) : Z(M_1) \longrightarrow Z(M_2).$$

Suppose we have another smooth manifold N_2 with boundary

$$\partial N_2 = M_2^{op} \sqcup M_3$$

corresponding to a linear map

$$Z(N_2) : Z(M_2) \longrightarrow Z(M_3).$$

We can then smoothly glue N_1 and N_2 along the common boundary component M_2 forming a new manifold

$$N = N_1 \cup_{M_2} N_2.$$

See, for example, [4]. Clearly

$$\partial N = M_1^{op} \sqcup M_3,$$

and N gives a cobordism from M_1 to M_3. The Atiyah-Segal sewing axiom [4] asserts that

$$
(2.1) \qquad
\begin{array}{ccc}
Z(M_1) \xrightarrow{Z(N_1)} Z(M_2) \xrightarrow{Z(N_2)} Z(M_3) \\
\| \downarrow \qquad\qquad\qquad\qquad \downarrow \| \\
Z(M_1) \xrightarrow{\quad Z(N) \quad} Z(M_3).
\end{array}
$$

A d-dimensional TQFT Z defines a topological invariant $Z(N)$ for each closed d-manifold N. The essence of TQFT is to break N into simpler pieces by cutting along $(d-1)$-dimensional submanifolds, and represent the invariant $Z(N)$ from that of simpler pieces. The situation is special for the case of 2D TQFT. First of all, we already know *all* topological invariants of a closed surface. They are just functions of the genus g of the surface. So there should be no reason for another theory to study them. Since there is only one 1-dimensional connected compact smooth manifold, a 2D TQFT is based on a single vector space

$$
(2.2) \qquad\qquad\qquad\qquad Z(S^1) = A,
$$

and its tensor products. What else could we gain?

The biggest surprise is that **a 2D TQFT is further quantized** [15, 25, 49, 67, 72]. This process starts with a Frobenius algebra A of (0.2). It is quantized to a quantum cohomology, and then further to Gromov-Witten invariants. Construction of spectral curves, and then quantizing them, have a parallelism to this story. This is the story we now explore in this and the next sections, but only to take a snapshot of the *converse* direction, i.e., from a quantum theory to a classical theory.

Let us denote by $\Sigma_{g,\bar{m},n}$ a connected, bordered, oriented, smooth surface of genus g with $m+n$ boundary circles. This is a surface obtained by removing $m+n$ disjoint open discs from a compact oriented two-dimensional smooth manifold of genus g without boundary. We assume that the boundary of $\Sigma_{g,\bar{m},n}$ itself is a smooth manifold, hence it is a disjoint union of $m+n$ circles. We give the induced orientation from the surface to n boundary circles, and the opposite orientation to the m boundary circles. The surface $\Sigma_{g,\bar{m},n}$ gives a cobordism of m circles to n circles. The orientation-preserving diffeomorphism class of such a surface is determined by the genus g and the two numbers m and n of boundary circles with different orientations. Therefore, the oriented equivalence class of cobordism is also determined by (g,m,n). The TQFT functor Z then assigns to each cobordism $\Sigma_{g,\bar{m},n}$ a multilinear map

$$
\omega_{g,\bar{m},n} \stackrel{\text{def}}{=} Z(\Sigma_{g,\bar{m},n}) : A^{\otimes m} \longrightarrow A^{\otimes n},
$$

which is completely determined by the label (g,m,n). This is the special situation of the two-dimensionality of TQFT, reflecting the simple topological classification of surfaces.

Suppose we have another cobordism $\Sigma_{h,\bar{k},m}$ of genus h from k circles to m circles, with orientation on the m circles induced from the surface, and the opposite orientation on k circles. We can *compose* two cobordisms, sewing the m circles of the first cobordism with the m circles of the second [65]. Here, we notice that on each pair of circles, one from the first surface and the other from the second, the orientations are the same, and hence we can put one on top of the other. Therefore, the orientations of the two surfaces are consistent after sewing. This sewing process generates a new surface $\Sigma_{g+h+m-1,\bar{k},n}$ of genus $g+h+m-1$. This is because the m pairs of circles sewn together create $m-1$ handles (see Figure 2.1). The **sewing axiom** of Atiyah-Segal [4] then requires that the functor Z associates the composition of linear maps to the composition of cobordisms. In our

situation, the composition of two maps generates a new map

$$\omega_{g,\bar{m},n} \circ \omega_{h,\bar{k},m} = \omega_{g+h+m-1,\bar{k},n} : A^{\otimes k} \longrightarrow A^{\otimes n},$$

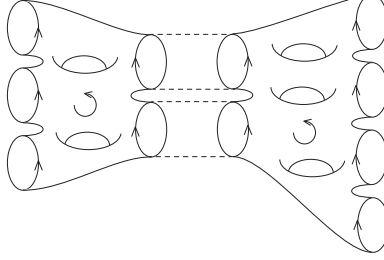corresponding to the sewing of cobordisms.

FIGURE 2.1. A string of **String Interactions**. The physics of a simple model for interacting strings is captured by the sewing axiom of Atiyah-Segal.

The sewing procedure can be generalized, allowing partial sewing of the boundary circles of matching orientation. For example, if $j \leq \ell$ and $j \leq m$, then gluing only j pairs of circles, we have a composition

(2.3) $$\omega_{g,\bar{m},n} \circ \omega_{h,\bar{k},\ell} = \omega_{g+h+j-1,\overline{k+m-j},n+\ell-j} : A^{\otimes k+m-j} \longrightarrow A^{\otimes n+\ell-j}.$$

Furthermore, since TQFT does not require cobordism to be given by a *connected* manifold, we can stack up disjoint union of surfaces and apply partial sewing to create a variety of linear maps from $A^{\otimes n}$ to $A^{\otimes m}$.

It was Dijkgraaf [17] who noticed the equivalence between 2D TQFTs and commutative Frobenius algebras. We can see the resemblance immediately. On the vector space A, we have operations defined by

(2.4)
$$\mathbf{1} = \omega_{0,\bar{0},1} : K \longrightarrow A, \qquad m = \omega_{0,\bar{2},1} : A^{\otimes 2} \longrightarrow A,$$
$$\epsilon = \omega_{0,\bar{1},0} : A \longrightarrow K, \qquad \delta = \omega_{0,\bar{1},2} : A \longrightarrow A^{\otimes 2},$$
$$\eta = \omega_{0,\bar{2},0} : A^{\otimes 2} \longrightarrow K.$$

A connected cobordism of m incoming circles and n outgoing circles is classified by the genus g of the surface. It does not depend on the history of how the cobordisms are glued together. Thus the diffeomorphism classes of surfaces after sewing cobordisms tell us relations among the operations of (2.4). For example,

$$\omega_{0,\bar{1},0} \circ \omega_{0,\bar{2},1} = \omega_{0,\bar{2},0} \Longrightarrow \epsilon \circ m = \eta,$$

which is (1.2). Associativity of multiplication comes from uniqueness of the topology of $\Sigma_{0,\bar{3},1}$ with one boundary circle carrying the induced orientation and three the opposite orientation. Changing the orientation of each boundary component to its opposite provides coassociativity. In this way $A = Z(S^1)$ acquires a bialgebra structure. To assure duality between algebra and coalgebra structures, we need to impose another condition here: non-degeneracy of $\eta = \omega_{0,\bar{2},0}$. Then A becomes a commutative Frobenius algebra.

Conversely, if we start with a finite-dimensional commutative Frobenius algebra A, then we first construct $\omega_{0,\bar{m},n}$ on the list of (2.4). More general maps $\omega_{g,\bar{m},n}$ are constructed by partial sewing (2.3). By construction, all these maps are associated with cobordism of circles, and hence determine a 2D TQFT [1, 65].

With the above considerations, we give the following definition.

Definition 2.1 (Two-dimensional Topological Quantum Field Theory)**.** Let (A, ϵ, λ) be a set of data consisting of a finite-dimensional vector space A over K, a non-trivial linear map $\epsilon : A \longrightarrow K$, and an isomorphism $\lambda : A \xrightarrow{\sim} A^*$. A **Two-dimensional Topological Quantum Field Theory** is a system $(A, \{\omega_{g,\bar{m},n}\})$ consisting of linear maps

$$\omega_{g,\bar{k},\ell} : A^{\otimes k} \longrightarrow A^{\otimes \ell}, \qquad 0 \le g, k, \ell,$$

satisfying the following axioms.

- **TQFT 1. Symmetry:**

(2.5)
$$\omega_{g,\bar{k},\ell} : A^{\otimes k} \longrightarrow A^{\otimes \ell}$$

   is symmetric with respect to the S_k-action on the domain.

- **TQFT 2. Non-triviality:**

(2.6)
$$\omega_{0,\bar{1},0} = \epsilon : A \longrightarrow K.$$

- **TQFT 3. Duality:**

(2.7)
$$
\begin{array}{ccc}
A^{\otimes m} & \xrightarrow{\;\;\omega_{g,\bar{m},n}\;\;} & A^{\otimes n} \\
\Big\downarrow{\scriptstyle \lambda^{\otimes m}} & & \Big\downarrow{\scriptstyle \lambda^{\otimes n}} \\
(A^*)^{\otimes m} & \xrightarrow{\;\;(\omega_{g,\bar{n},m})^*\;\;} & (A^*)^{\otimes n}
\end{array}.
$$

- **TQFT 4. Sewing:**

(2.8) $\quad \omega_{g,\bar{m},n} \circ \omega_{h,\bar{k},\ell} = \omega_{g+h+j-1,\overline{k+m-j},n+\ell-j} : A^{\otimes k+m-j} \longrightarrow A^{\otimes n+\ell-j}, \quad j \le m, \; j \le \ell.$

Remark 2.2. As noted above, the vector space A acquires the structure of a commutative Frobenius algebra from these axioms. Conversely, if A is a commutative Frobenius algebra, then $\omega_{0,\bar{m},n}$ of (2.4) can be extended to $\omega_{g,\bar{m},n}$ that satisfy the TQFT axioms.

3. Silhouette of Gromov-Witten theory

Recall that for any oriented closed manifold X, its cohomology ring $H^*(X, K)$ is a Frobenius algebra defined over K. If we restrict ourselves to even dimensional manifold X and consider only even part of the cohomology, then $A = H^{even}(X, K)$ is a finite-dimensional commutative Frobenius algebra. This Frobenius algebra naturally defines a 2D TQFT. The role of a TQFT in general dimension is to represent a topological invariant of higher dimensional manifolds. We are now seeing that the *classical* topology of a manifold X is a 2D TQFT. Changing our point of view, we can ask if we start with X, then how do we find the corresponding 2D TQFT? Of course the Frobenius algebra structure automatically determines the unique 2D TQFT as we have seen in Section 2. But then the picture of sewing cobordisms is lost in this algebraic formulation. What is the role of the cobordism of circles in the context of understanding the manifold X?

The amazing vision emerged in the early 1990s is that 2D TQFT can be further *quantized* into **Gromov-Witten theory**, which produces *quantum* topological invariants of X that may carry more information than classical invariants. This epoch making discovery was one of the decisive moments of the fruitful interaction of string theory and geometry, which is still continuing today.

String theory deals with flying strings in a space-time manifold X. The trajectory of a moving string is a curved cylinder embedded in the manifold X, like a duct pipe in the attic. When strings are considered as *quantum* objects, they can interact one another. For example, Figure 2.1 can be interpreted as a *string* of **string interactions**. First, three

strings collide in a complicated interaction to produce two strings. The complexity of the interaction is classified as "$g = 2$" in terms of the genus of the first surface. These two strings created in the first interaction then collide in an even more complicated (i.e., $g = 3$) interaction to produce four strings in the end. The trajectories of these interactions form a bordered surface of genus 6 with 7 boundary components embedded in the space-time X.

Models for string theory were originally introduced to understand *Regge trajectories*, the poles of the scattering amplitude of a quantum system as a function in energy and angular momentum being considered as *complex variables*, in the context of quark confinement [89]. This complex analytic nature of string theory was further extended in the discovery [51] that geometry of the space-time X dictates possible string interactions. In this process, the importance of Calabi-Yau spaces was recognized, through considerations on the consistency of string theory, known as *anomaly cancellation*, as a physical theory. By changing the point of view 180°, researchers then noticed that a simple model of string interactions could be used to obtain a totally new class of topological invariants of X itself. These new invariants are not necessarily captured by the classical topology, such as (co)homology groups and homotopy groups. We use the terminology of **quantum topological invariants** for the invariants obtained by string theory considerations.

Gromov-Witten theory is a mathematically rigorous interpretation of a model for a quantum string theory. A trajectory of interacting strings in X is the image of a map

(3.1) $$f : \Sigma_{g,\bar{m},n} \longrightarrow X$$

from a bordered surface $\Sigma_{g,\bar{m},n}$ into X. Recall that the fundamental group $\pi_1(X, x_0)$ of X is determined by the connectivity of the pointed loop space

$$\mathcal{L}(X, x_0) = \mathcal{C}^\infty\big((S^1, *), (X, x_0)\big),$$

which is the *moduli space* of differentiable maps from S^1 to X that send a reference point $* \in S^1$ to $x_0 \in X$. In the same spirit, Gromov-Witten invariants are defined by looking at the *classical* topological invariants of appropriate moduli spaces of maps from $\Sigma_{g,\bar{m},n}$ to X. When we have the notion of *size* of a string, the trajectory $\Sigma_{g,\bar{m},n}$ has a metric on it. If the ambient manifold X has a geometric structure, such as a symplectic structure, complex structure, or a Kähler structure, then the map f needs to be compatible with the structures of the source and the target. For example, if X is Kähler, then we give a complex structure on $\Sigma_{g,\bar{m},n}$ that is compatible with the metric it has, and require that f is a holomorphic map. Technical difficulties arise in defining moduli spaces of such maps. Even the moduli spaces are defined, they are often very different objects from the usual differentiable manifolds. We need to extend the notion of manifolds here. Thus identifying reasonable classical topological invariants of these spaces also poses a difficult problem.

The simplest scenario is the following: We take $X = pt$ to be just a single point. Of course nobody needs to know the topological structure of a point. We all know it! By exploring the Gromov-Witten theory of a point, we learn the structure of the theory itself. Since we can map *anything* to a point, a point may not be such a simple object, after all. It is like considering a *vacuum* in physics. Again, anything can be thrown into a vacuum. A quantum theory of a vacuum is inevitably a rich theory.

When $X = pt$ is a point, we consider all incoming and outgoing strings to be infinitesimally small. Thus the surfaces we are considering become closed, and boundary components are just several points identified on them. By considering its conformal class, a metric on a closed surface naturally gives rise to a unique complex structure, making it a compact *Riemann surface*. And every compact Riemann surface acquires a unique *projective algebraic* structure, becoming a projective algebraic curve. Since we are allowing strings to shrink

infinitesimally small during the interaction, the trajectory may contain a process of an embedded S^1 in the surface shrinking to a point, and then expanding into a finite size circle again a moment later. Such a process can be understood in complex algebraic geometry as a **nodal singularity** of an algebraic curve. Locally, every nodal singularity of a curve is the same as the neighborhood of the origin of a singular algebraic curve

$$\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}.$$

We are talking about an abstract notion of trajectories here, because nothing can move in $X = pt$. In terms of the map idea (3.1), we note that there is a *unique* map f from any projective algebraic curve to a point, which satisfies any reasonable requirement of maps such as being a morphism of algebraic varieties. Then the moduli space of all maps simply means the moduli space of the source.

A **stable curve** [67] is a projective algebraic curve with only nodal singularities and a finite number of smooth points marked on the curve such that it has only a finite number of algebraic automorphisms fixing each marked point. We recall that the holomorphic automorphism group of $\mathbb{P}^1$,

$$\mathrm{Aut}(\mathbb{P}^1) = PSL_2(\mathbb{C}),$$

acts on $\mathbb{P}^1$ triply transitively. The number of automorphisms fixing up to 2 points is infinity. Since $PSL_2(\mathbb{C})$ is compact and three-dimensional, if we choose three distinct points on $\mathbb{P}^1$ and require the automorphism to fix each of these points, then we have only finitely many choices. Actually in our case, it is unique. For an elliptic curve E, since it is an abelian group, E acts on E transitively. To avoid these translations, we need to choose a point on E. We can naturally identify it as the identity element of the elliptic curve as a group. Automorphisms of an elliptic curve fixing a point then form a finite group. If a compact Riemann surface C has genus $g = g(C) \geq 2$, then it is known that the order of the analytic automorphism group is bounded by

$$|\mathrm{Aut}(C)| \leq 84(g - 1).$$

This bound comes from hyperbolic geometry. It is easy to see the finiteness. First, we note that universal covering of C is the upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\},$$

and C is constructed by the quotient

$$C \xrightarrow{\sim} \mathbb{H}/\rho(\pi_1(C))$$

through a faithful representation

(3.2) $$\rho : \pi_1(C) \longrightarrow PSL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$$

of the fundamental group of C into the automorphism group of $\mathbb{H}$. Every holomorphic automorphism of C extends to an automorphism of $\mathbb{H}$. We know that C does not have any non-trivial holomorphic vector field v. If it did, then v would be a differentiable vector field with isolated zeros, and each zero comes with positive index, because locally it is given by $z^n d/dz$. We learn from topology that the sum of the indices of isolated zeros of a vector field on C is equal to $\chi(C) = 2 - 2g < 0$. It is a contradiction. Thus $\mathrm{Aut}(C) \subset PSL_2(C)$ is a discrete subgroup. Since $PSL_2(\mathbb{R})$ is compact, $|\mathrm{Aut}(C)|$ is finite.

An algebraic curve C is stable if and only if (1) every singularity is nodal, and (2) every irreducible component C' of C has a finite number of automorphisms. The second condition means that the total number of smooth marked points and singular points of C that are on

C' has to be 3 or more if $g(C') = 0$, and 1 or more if $g(C') = 1$. There is no condition for an irreducible component of genus two or more.

For a pair (g, n) of integers $g \geq 0$ and $n \geq 1$ in the stable range $2g - 2 + n > 0$, we denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable curves of genus g and n smooth marked points. It is a complex orbifold of dimension $3g - 3 + n$. Quantum topological invariants of a point $X = pt$ are then realized as classical topological invariants of $\overline{\mathcal{M}}_{g,n}$. Alas to this day, we cannot identify the cohomology groups $H^i(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ for all values of i, g, n. From the very definition of these moduli spaces, we can construct many concrete cohomology classes on each of $\overline{\mathcal{M}}_{g,n}$, called *tautological* classes. What we do not know is indeed how much more we need to know to determine all of $H^i(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. The surprise of Witten's conjecture [104], proved in [66], is that we can actually explicitly write intersection relations of certain tautological classes for *all* values of g and n.

Among many different proofs available for the Witten conjecture (see for example, [64, 66, 76, 77, 86, 92]), [77, 86] deal with recursive relations among surfaces much in the same spirit of TQFT. These relations are first noticed in [21]. We will discuss these relations in connection to topological recursion later in these lectures.

Geometric relations among $\overline{\mathcal{M}}_{g,n}$'s for different values of (g, n) have a simple meaning. Let us denote by $\mathcal{M}_{g,n}$ the moduli space of smooth n-pointed curves. Then the boundary

$$\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$$

consists of points representing singular curves. Simplest singular stable curve has one nodal singularity, which can be described as collision of two smooth points. Analyzing how these singularities occur via degeneration, we come up with three types of natural morphisms among the moduli spaces $\overline{\mathcal{M}}_{g,n}$. They are the forgetful morphisms

$$(3.3) \qquad\qquad \pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

which simply erase one of the marked points on a stable curve, and gluing morphisms

$$(3.4) \qquad\qquad gl_1 : \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$$(3.5) \qquad\qquad gl_2 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

that construct boundary strata of $\overline{\mathcal{M}}_{g,n}$. Under a gluing morphism, we put two smooth points of stable curves together to form a one nodal singularity. The first one gl_1 glues two points on the same curve together, and gl_2 one each on two curves.

The fiber of π at a stable curve $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ is the curve C itself, because another marked point can be placed anywhere on C. Thus π is a *universal family* of curves parameterized by the base moduli space $\overline{\mathcal{M}}_{g,n}$. Let us examine the fiber more carefully. When we place an extra marked point p_{n+1} at p_i, $i = 1, \ldots, n$, the data represented by this point of $\overline{\mathcal{M}}_{g,n+1}$ on the fiber of π is a singular curve that is obtained by joining C itself with a $\mathbb{P}^1$ at the location of $p_i \in C$, but the two marked points p_i, p_{n+1} are actually placed on the line $\mathbb{P}^1$. Assigning this singular curve to $(C, p_1, \ldots, p_n)$ defines a section

$$\sigma_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n+1},$$

which is a right inverse of π. Geometrically, σ_i sends $(C, p_1, \ldots, p_n)$ to the point p_i on the fiber $C = \pi^{-1}(C, p_1, \ldots, p_n)$. Obviously,

$$\pi \circ \sigma_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

is the identity map. This is because erasing p_{n+1} causes the $\mathbb{P}^1$ component becoming unstable, hence π corresponds to the operation of eliminating the component altogether and placing p_i at the location of $\mathbb{P}^1$. This process is called *stabilization.* The description

of the fiber over $(C, p_1, \ldots, p_n)$ with a *singular* stable curve C is similar, when we try to choose the location of p_{n+1} at a singular point of C. We first locally normalize C near at the nodal singularity. Thus we have two points of the local normalization that correspond to the original singular point. We then connect these two points by a line $\mathbb{P}^1$. Finally, we place p_{n+1} as the third point on $\mathbb{P}^1$. Therefore, when the image of the section σ_i hits a singular point of the fiber, the image point represents a singular curve that has exactly one more singular point. The forgetful morphism erases p_{n+1}, but then the process leaves only two points on the $\mathbb{P}^1$ component, which is unstable. We appeal to the stabilization again.

Gromov-Witten theory for a Kähler manifold X [15] concerns topological structure of the moduli space

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \left\{ f : (C, p_1, \ldots, p_n) \longrightarrow X \mid [f(C)] = \beta \right\}$$

of *stable* holomorphic maps f from a nodal curve C with n smooth marked points $p_1, \ldots, p_n \in C$ to X such that the homology class of the image $f(C)$ agrees with a prescribed homology class $\beta \in H_2(X, \mathbb{Z})$. Here, stability of f is defined by imposing the finiteness of possible automorphisms. Giving a definition of this moduli space is beyond our scope of this article. We refer to [15]. The moduli space, once defined, should come with natural maps

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \xrightarrow{\; ev_i \;} X$$
$$\phi \downarrow$$
$$\overline{\mathcal{M}}_{g,n},$$

where the forgetful map ϕ assigns the stabilization of the source $(C, p_1, \ldots, p_n)$ to the map f by forgetting about the map itself, and

$$ev_i\big(f : (C, p_1, \ldots, p_n) \longrightarrow X \big) = f(p_i) \in X, \qquad i = 1, \ldots, n,$$

is the value of f at the i-th marked point $p_i \in C$. Stabilization means that every irreducible component of $(C, p_1, \ldots, p_n)$ that is not stable is shrunk to a point. It is necessary because a stable map f may be defined on an unstable source. If we indeed know the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and that its cohomology theory behaves as we expect, then we would have

$$H^*\big(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}\big) \xleftarrow{\; ev_i^* \;} H^*(X, \mathbb{Q})$$
$$\phi_! \downarrow$$
$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

where $\phi_!$ is the *Gysin map* defined by integration along fiber. If $\overline{\mathcal{M}}_{g,n}(X, \beta)$ were a manifold, and the map $\phi : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{g,n}$ were a fiber bundle, then $\overline{\mathcal{M}}_{g,n}(X, \beta)$ would have been locally a direct product, hence the Gysin map $\phi_!$ associated with ϕ would be defined by integrating de Rham cohomology classes of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ along fiber of ϕ. Choose any cohomology classes $v_1, \ldots v_n \in H^*(X, \mathbb{Q})$ of X. We then could have defined the **Gromov-Witten invariants** by

$$GW_{g,n}^{X, \beta}(v_1, \ldots, v_n) := \int_{\overline{\mathcal{M}}_{g,n}} \phi_! \big(ev_1^*(v_1) \cdots ev^*(v_n) \big).$$

However, the general construction of the Gysin map does not work as we hope. This is due to the complicated nature of the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$, which often has components of *unexpected* dimensions. The remedy is to define the *virtual fundamental class*

$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ of the expected dimension, and avoid the use of $\phi_!$ by defining

$$(3.6) \qquad GW_{g,n}^{X,\beta}(v_1,\ldots,v_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(v_1)\cdots ev^*(v_n).$$

We refer to [15, 72] for more detail.

$A = H^{even}(X,\mathbb{Q})$ is a Frobenius algebra. Although Gromov-Witten theory goes through a big black box $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$, what we wish is a map

$$\Omega_{g,n} : H^{even}(X,\mathbb{Q})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

whose integral over $\overline{\mathcal{M}}_{g,n}$ gives quantum invariants of X. Then what are the properties that Gromov-Witten invariants should satisfy? Can we list the properties as axioms for the above map $\Omega_{g,n}$ so that we can characterize Gromov-Witten invariants? This was one of the motivations of Kontsevich and Manin to introduce *cohomological field theory* in [67, 72]. As we see below, a 2D TQFT can be obtained as a special case of a CohFT. The amazing relation between TQFT and CohFT, i.e., the *reconstruction* of CohFT from its restriction to TQFT due to Givental and Teleman [49, 101], plays a key role in many new developments (see for example, [3, 37, 40, 73]), some of which are deeply related with topological recursion.

4. COHOMOLOGICAL FIELD THEORY

The idea of moduli spaces of algebraic curves was introduced by Riemann [96] in 1857 as a parameter space of birational equivalence classes of algebraic curves. The use of the word **moduli** is due to him. He also noticed that a point of this space can be realized as a Riemann period matrix. A characterization of period matrices among symmetric matrices can be obtained by an infinite-dimensional integrable system of non-linear partial differential equations (see for example [79]). The deformation theory of Kodaira and Spencer of the 1950s gave a definition of *infinitesimal* moduli spaces in any dimensions. The modern concept of *global* moduli of curves is culminated in [16], which identifies $\overline{\mathcal{M}}_g$ as a **stack**. Once this concept is established, it becomes possible to ask its global topology. Yet it has taken a few more decades for mathematics community to arrive at the idea that the moduli of *pointed* curves form a network, which provides inductive mechanisms to investigate topological structures of $\overline{\mathcal{M}}_{g,n}$ [54, 67, 72, 87]. In this context, extremely surprisingly, an infinite-dimensional integrable system of the same nature appears again [66, 104].

Following Pixton [95], let us introduce **stable graphs**. This naming is an abbreviation of "dual graphs of stable curves." A stable graph $\gamma = (V, E, L)$ is a connected graph consisting of a set V of vertices, a set E of edges, and a set L of *legs*, with prescribed incidence relations among them. Every edge is incident to two vertices, which can be the same, i.e., an edge can be a loop. Two vertices can be connected by two or more edges. Each vertex $v \in V$ carries a non-negative integer $g(v)$ called the genus of v. A leg is a line segment attached to a vertex. We do not consider the other end of a leg as a vertex. The sum of the number of legs attached to a vertex v and the degree of v, i.e., the number of half-edges incident to v, is denoted by $n(v)$. The stability condition we impose is that at each vertex, we have

$$(4.1) \qquad 2g(v) - 2 + n(v) > 0.$$

The graph γ is a 1-dimensional topological space. We denote by $h_i(\gamma)$, $i = 0, 1$, the dimension of the homology group $H_i(\gamma,\mathbb{Q})$. A stable graph γ is of *type* (g,n) if

$$(4.2) \qquad \begin{cases} g = \sum_{v\in V} g(v) + h_1(\gamma) \\ n = |L|. \end{cases}$$

A stable graph represents a stable curve of arithmetic genus g with n marked points, consisting of $|V|$ irreducible components of geometric genus $g(v)$ for each $v \in V$ connected by $|E|$ nodal singularities. A leg attached v indicates on which component this marked point is placed. An edge between $v, w \in V$ describes a nodal singularity, which is the intersection of the components corresponding to these two vertices. A loop attached to a single vertex represents a self-intersection of the component.
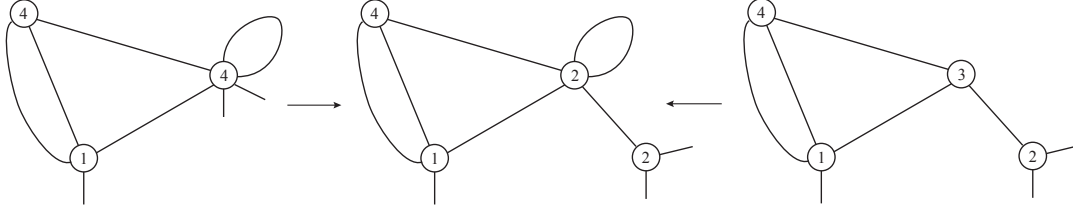

FIGURE 4.1. Stable graphs of type $(12, 3)$. The arrows show degeneration of stable curves. The reverse direction of an arrow is an edge-contraction operation. When a loop is contracted at a vertex, the genus of the vertex is increased by 1. When an edge in between two vertices is contracted, the genera of the two vertices are added together.

A smooth curve of genus g with n marked points is represented by a stable graph of one vertex of genus g and n legs attached to it. Since there is no singularity on this curve, the stable graph has no edge. The number of edges $|E|$ shows the codimension of the boundary stratum of $\overline{\mathcal{M}}_{g,n}$ on which the stable curve belongs. Thus the maximum number of edges on a stable graph of type (g, n) is $3g - 3 + n$. The Euler characteristic of γ is

$$h_0(\gamma) - h_1(\gamma) = 1 - g.$$

Therefore, the maximum number of vertices of a stable graph is $2g - 2 + n$, in which case every vertex $v \in V$ carries genus 0 and $n(v) = 3$.
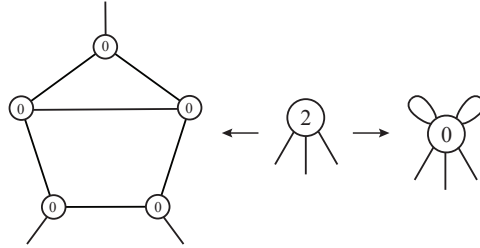
FIGURE 4.2. Stable graphs of type $(2, 3)$, showing two degenerations of a smooth curve to singular stable curves consisting of components of geometric genus 0.

The network of stable graphs of the same type is connected by *edge-contraction operations*. In the next section we will study edge contractions for cell graphs. Even as the same operation called edge contraction, the nature of this operation for a cell graph is quite different from stable graphs. In a stable graph, an edge represents a nodal singularity of a stable curve. Thus contracting this edge is a process in the reverse direction of the degeneration of a curve forming a nodal singularity. As shown in Figure 4.1, when we contract an edge connecting two vertices v and w of a stable graph $\gamma = (V, E, L)$, the genus of the vertex becomes $g(v) + g(w)$. When we contract a loop attached to v, the genus goes up by

1 at v. If we contract all edges of a stable graph of type (g, n), then we obtain the graph with a single vertex of genus g and all n legs attached to it. Thus the network of stable graphs of the same type is connected by edge-contraction operations.

Definition 4.1 (Cohomological Field Theory [67, 72]). Let A be a finite-dimensional, unital, associative, and commutative Frobenius algebra with a basis $\{e_1, \ldots, e_r\}$. A **Cohomological Field Theory** is a system $(A, \{\Omega_{g,n}\})$ consisting of linear maps

$$(4.3) \qquad\qquad \Omega_{g,n} : A^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, K)$$

defined for (g, n) in the stable range $2g - 2 + n > 0$ and satisfying the following axioms:

CohFT 0: $\Omega_{g,n}$ is S_n-invariant, and $\Omega_{0,3}(v_1, v_2, v_3) = \eta(v_1 v_2, v_3)$.

CohFT 1: $\Omega_{g,n+1}(v_1, \ldots, v_n, \mathbf{1}) = \pi^* \Omega_{g,n}(v_1, \ldots, v_n)$.

CohFT 2: $gl_1^* \Omega_{g,n}(v_1, \ldots, v_n) = \sum_{a,b} \Omega_{g-1,n+2}(v_1, \ldots, v_n, e_a, e_b) \eta^{ab}$.

CohFT 3: $gl_2^* \Omega_{g_1+g_2, |I|+|J|}(v_I, v_J) = \sum_{a,b} \eta^{ab} \Omega_{g_1, |I|+1}(v_I, e_a) \otimes \Omega_{g_2, |J|+1}(v_J, e_b),$

where $I \sqcup J = \{1, \ldots, n\}$ is a disjoint partition of the index set, and the tensor product operation on the right-hand side is performed via the Künneth formula of cohomology rings

$$H^*(\overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, n_2+1}, K) \cong H^*(\overline{\mathcal{M}}_{g_1, n_1+1}, K) \otimes H^*(\overline{\mathcal{M}}_{g_2, n_2+1}, K).$$

Remark 4.2. The condition $\Omega_{0,3}(v_1, v_2, v_3) = \eta(v_1 v_2, v_3)$ says that the product of the Frobenius algebra is determined by the $(0, 3)$-value of the CohFT.

Remark 4.3. The condition for gluing morphisms says the following: The assignment of $\mathbf{1}$ to a singular point after gluing is identified with assigning $\delta(\mathbf{1})$ to the two normalization points via (1.11). In terms of stable graphs, the formulas read as follows. We consider a stable graph of type (g, n) with exactly one edge. First we assign $\mathbf{1}$ to this edge. Next, we *cut* it to create two extra legs. This process corresponds to the normalization at the nodal singularity represented by the edge. Then we assign $\delta(\mathbf{1})$ to these two legs, and identify the pull-back of $\Omega_{g,n}$ with the value coming from the corresponding boundary strata to the stable graph.

Proposition 4.4 (2D TQFT is a CohFT). *Every 2D TQFT is a CohFT that takes values in $H^0(\overline{\mathcal{M}}_{g,n}, K)$. More precisely, let $(A, \omega_{g,\bar{m},n})$ be a 2D TQFT. Then $\omega_{g,n} = \omega_{g,\bar{n},0}$ satisfies the CohFT axioms, by identifying $K = H^0(\overline{\mathcal{M}}_{g,n}, K)$.*

Proof. Since $\overline{\mathcal{M}}_{g,n}$ is connected, the three types of morphisms (3.3), (3.4), and (3.5) all produce isomorphisms of degree 0 cohomologies. Thus the axioms CohFT 1–3 become

$$(4.4) \qquad \omega_{g,n+1}(v_1, \ldots, v_n, \mathbf{1}) = \omega_{g,n}(v_1, \ldots, v_n),$$

$$(4.5) \qquad \omega_{g,n}(v_1, \ldots, v_n) = \sum_{a,b} \omega_{g-1,n+2}(v_1, \ldots, v_n, e_a, e_b) \eta^{ab},$$

$$(4.6) \qquad \omega_{g_1+g_2, |I|+|J|}(v_I, v_J) = \sum_{a,b} \eta^{ab} \omega_{g_1, |I|+1}(v_I, e_a) \cdot \omega_{g_2, |J|+1}(v_J, e_b).$$

We wish to show that (4.4)-(4.6) are consequences of the partial sewing axiom (2.8) of TQFT under the identification $\omega_{g,n} = \omega_{g,\bar{n},0}$.

Since $\omega_{0,\bar{0},1} = \mathbf{1}$, we have

$$\omega_{g, \overline{n+1}, 0} \circ_{(n+1)} \omega_{0,\bar{0},1} = \omega_{g,\bar{n},0} = \omega_{g,n} : A^{\otimes n} \longrightarrow K$$

from (2.8), which is (4.4). Here, $\circ_{(n+1)}$ is the composition taken at the $(n+1)$-th slot of the input variables of $\omega_{g,\overline{n+1},0}$. Next, we need to identify $\omega_{0,\bar{0},2} : K \longrightarrow A \otimes A$. Since $\omega_{0,\bar{0},2} = \omega_{0,\bar{1},2} \circ \omega_{0,\bar{0},1}$, we see that $\omega_{0,\bar{0},2}(1) = \delta(\mathbf{1}) = \sum_{a,b} \eta^{ab} e_a \otimes e_b$. Denoting by $\circ_{(n+1,n+2)}$ to indicate composition taking at the last two slots of variables, we have (4.5)

$$\omega_{g-1,\overline{n+2},0} \circ_{(n+1,n+2)} \omega_{0,\bar{0},2} = \omega_{g,\bar{n},0} : A^{\otimes n} \longrightarrow K.$$

If we have two disjoint sets of variables v_I and v_J, then we can apply composition of $\omega_{0,\bar{0},2}$ simultaneously to two different maps. For example, we have

$$\left(\omega_{g_1,\overline{|I|+1},0} \otimes \omega_{g_2,\overline{|J|+1},0} \right) \circ \omega_{0,\bar{0},2} = \omega_{g_1+g_2,\overline{|I|+|J|},0} : A^{\otimes(|I|+|J|)} \longrightarrow K,$$

which is (4.6).

Notice that a linear map

$$\omega_{g,m+n} : A^{\otimes m} \otimes A^{\otimes n} \longrightarrow K$$

is equivalent to $A^{\otimes m} \longrightarrow (A^*)^{\otimes n}$. Thus we can re-construct a map $\omega_{g,\bar{m},n}$ from $\omega_{g,m+n}$ by

(4.7)
$$
\begin{array}{ccc}
A^{\otimes m} \xrightarrow{\ \omega_{g,m+n}\ } (A^*)^{\otimes n} \xrightarrow{\ (\lambda^{-1})^{\otimes n}\ } A^{\otimes n} \\
\| \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \| \ , \\
A^{\otimes m} \xrightarrow{\qquad\qquad \omega_{g,\bar{m},n} \qquad\qquad} A^{\otimes n}
\end{array}
$$

where $\lambda : A \xrightarrow{\ \sim\ } A^*$ is the isomorphism of (1.4). For the case of $g = 0$, $m = 2$, and $n = 1$, (4.7) implies that

$$\lambda^{-1}\omega_{0,3}(v_1, v_2,\ \cdot\ ) = \omega_{0,\bar{2},1}(v_1, v_2) = v_1 v_2$$

for $v_1, v_2 \in A$. Or equivalently, we have

(4.8) $$\omega_{0,3}(v_1, v_2, v_3) = \eta(v_1 v_2, v_3) = \epsilon(v_1 v_2 v_3).$$

This completes the proof. $\square$

Conversely, the degree 0 part of the cohomology of a CohFT is a 2D TQFT.

Proposition 4.5 (Restriction of CohFT to the degree 0 part of the cohomology ring)**.** *Let* $(A, \Omega_{g,n})$ *be a CohFT associated with a Frobenius algebra* A. *The Frobenius algebra* A *itself defines a unique 2D TQFT* $(A, \omega_{g,\bar{m},n})$. *Denote by*

$$r : H^*(\overline{\mathcal{M}}_{g,n}, K) \longrightarrow H^0(\overline{\mathcal{M}}_{g,n}, K) = K$$

the restriction of the cohomology ring to its degree 0 *component, and define*

(4.9) $$\omega_{g,n} = r \circ \Omega_{g,n} : A^{\otimes n} \longrightarrow K.$$

Then we have the equality of maps

(4.10) $$\omega_{g,n} = \omega_{g,\bar{n},0} : A^{\otimes n} \longrightarrow K$$

for all (g, n) *with* $2g - 2 + n > 0$. *In other words, the degree* 0 *restriction of a CohFT is the 2D TQFT determined by the Frobenius algebra* A.

Proof. We already know that $(A, \omega_{g,\bar{m},n})$ defines a CohFT with values in $H^0(\overline{\mathcal{M}}_{g,n}, K)$. We need to show that this CohFT is exactly the degree 0 restriction of the *given* CohFT we start with. First we extend $\omega_{g,n}$ to the unstable range by

(4.11) $$\omega_{0,1} = \epsilon : A \longrightarrow K, \qquad \omega_{0,2} = \eta : A^{\otimes 2} \longrightarrow K.$$

We then note that from (2.4) and (4.11), we see that (4.10) holds for $\omega_{0,1}$ and $\omega_{0,2}$. The general case of (4.10) follows by induction on $3g - 3 + n$, provided that $\omega_{0,3}$ is appropriately

defined. This is because (4.5) and (4.6) are induction formula recursively generating $\omega_{g,n}$ from those with smaller values of g and n. Since

- Case of (4.5): $3g - 3 + n = [3(g-1) - 3 + (n+2)] + 1$,
- Case of (4.6): $3(g_1 + g_2) - 3 + |I| + |J| = [3g_1 - 3 + |I| + 1] + [3g_2 - 3 + |J| + 1] + 1$,

we see that the *complexity* $3g - 3 + n$ is always reduced by 1 in each of the recursive formulas.

Finally, we see that since $\overline{\mathcal{M}}_{0,3}$ is just a point as we have noted above in the discussion of $\mathrm{Aut}(\mathbb{P}^1)$, $\omega_{0,3} = \Omega_{0,3}$ holds. Hence

$$(4.12) \qquad \omega_{0,3}(v_1, v_2, v_3) = \eta(v_1 v_2, v_3),$$

which makes (4.10) holds for all values of (g, n). This completes the proof. $\qquad\square$

Remark 4.6. The above two propositions show that a 2D TQFT can be defined either just by a commutative Frobenius algebra A, by a system of maps $\{\omega_{g,\bar{m},n}\}$ satisfying the TQFT axioms, or the degree 0 part of a CohFT. From now on, we use $(A, \omega_{g,n})$ to denote a 2D TQFT, which is less cumbersome and easier to deal with.

Proposition 4.7. *The genus* 0 *values of a 2D TQFT is given by*

$$(4.13) \qquad \omega_{0,n}(v_1, \ldots, v_n) = \epsilon(v_1 \cdots v_n).$$

Proof. This is a direct consequence of CohFT 3 and (1.10). $\qquad\square$

One of the original motivations of TQFT [4, 99] is to identify the *topological invariant* $Z(N)$ of a closed manifold N. In our current setting, it is defined as

$$(4.14) \qquad Z(\Sigma_g) := \epsilon\big(\lambda^{-1}(\omega_{g,1})\big)$$

for a closed oriented surface Σ_g of genus g. Here, $\omega_{g,1} : A \longrightarrow K$ is an element of A^*, and $\lambda : A \xrightarrow{\sim} A^*$ is the canonical isomorphism (1.4).

Proposition 4.8. *The topological invariant* $Z(\Sigma_g)$ *of* (4.14) *is given by*

$$(4.15) \qquad Z(\Sigma_g) = \epsilon(\mathbf{e}^g),$$

where $\mathbf{e} \in A$ *is the Euler element of* (1.13).

Lemma 4.9. *We have*

$$(4.16) \qquad \mathbf{e} := m \circ \delta(1) = \lambda^{-1}(\omega_{1,1}).$$

Proof. This follows from

$$\omega_{1,1}(v) = \sum_{a,b} \omega_{0,3}(v, e_a, e_b)\eta^{ab} = \sum_{a,b} \eta(v, e_a e_b)\eta^{ab} = \eta(v, \mathbf{e})$$

for every $v \in A$. $\qquad\square$

Proof of Proposition 4.8. Since the starting case $g = 1$ follows from the above Lemma, we prove the formula by induction, which goes as follows:

$$
\begin{aligned}
\omega_{g,1}(v) &= \sum_{a,b} \omega_{g-1,3}(v, e_a, e_b)\eta^{ab} \\
&= \sum_{i,j,a,b} \omega_{0,4}(v, e_a, e_b, e_i)\omega_{g-1,1}(e_j)\eta^{ab}\eta^{ij} \\
&= \sum_{i,j,a,b} \eta(v e_a e_b, e_i)\omega_{g-1,1}(e_j)\eta^{ab}\eta^{ij}
\end{aligned}
$$

$$= \sum_{i,j} \eta(v\mathbf{e}, e_i)\omega_{g-1,1}(e_j)\eta^{ij}$$
$$= \omega_{g-1,1}(v\mathbf{e})$$
$$= \omega_{1,1}(v\mathbf{e}^{g-1})$$
$$= \eta(v\mathbf{e}^{g-1}, \mathbf{e}) = \eta(v, \mathbf{e}^g).$$

□

A closed genus g surface is obtained by sewing g genus 1 pieces with one output boundaries to a genus 0 surface with g input boundaries. Since the Euler element is the output of the genus 1 surface with one boundary, we obtain the same result

$$Z(\Sigma_g) = \omega_{0,g}(\overbrace{\mathbf{e}, \ldots, \mathbf{e}}^{g}).$$

Finally we have the following:

Theorem 4.10. *The value of a 2D TQFT is given by*

(4.17) $$\omega_{g,n}(v_1, \ldots, v_n) = \epsilon(v_1 \cdots v_n \mathbf{e}^g).$$

Proof. The argument is the same as the proof of Proposition 4.8:

$$\omega_{g,n}(v_1, \ldots, v_n) = \omega_{1,n}(v_1\mathbf{e}^{g-1}, v_2, \ldots, v_n)$$
$$= \sum_{a,b} \omega_{0,n+2}(v_1\mathbf{e}^{g-1}, v_2, \ldots, v_n, e_a, e_b)\eta^{ab}$$
$$= \epsilon(v_1 \cdots v_n \mathbf{e}^g).$$

□

Remark 4.11. Since $\overline{\mathcal{M}}_{g,n}$ is connected and the restriction of its degree 0 cohomology group to any boundary stratum is isomorphic, we can calculate $\omega_{g,n}$ by using any stable graph we like. For example, we can use the one on the right of Figure 4.2, which has a vertex of genus 0 with n legs and g loops. We assign $\mathbf{1}$ to every loop, and cut it at the midpoint. CohFT 2 and (4.13) then gives the same value as in (4.17).

5. Category of cell graphs

In the original formulation of 2D TQFT, the operations of multiplication and comultiplication are associated with an oriented surface of genus 0 with three boundary circles. In this section, we introduce a category of ribbon graphs, which carries the information of all finite-dimensional Frobenius algebras. To avoid unnecessary confusion, we use the terminology of *cell graphs* in this article, instead of more common ribbon graphs. Ribbon graphs naturally appear for encoding complex structures of a topological surface (see for example, [66, 81]). Our purpose of using ribbon graphs are for degeneration of stable curves, and we label vertices, instead of *faces*, of a ribbon graph.

Definition 5.1 (Cell graphs). A connected **cell graph** of topological type (g, n) is the 1-skeleton of a cell-decomposition of a connected closed oriented surface of genus g with n labeled 0-cells. We call a 0-cell a *vertex*, a 1-cell an *edge*, and a 2-cell a *face*, of the cell graph. We denote by $\Gamma_{g,n}$ the set of connected cell graphs of type (g, n). Each edge consists of two **half-edges** connected at the midpoint of the edge.

Remark 5.2.   • The *dual* of a cell graph is a ribbon graph, or Grothendieck's dessin
d'enfant. We note that we label vertices of a cell graph, which corresponds to face
labeling of a ribbon graph. Ribbon graphs are also called by different names, such
as embedded graphs and maps.
   • We identify two cell graphs if there is a homeomorphism of the surfaces that brings
one cell-decomposition to the other, keeping the labeling of 0-cells. The only possible
automorphisms of a cell graph come from cyclic rotations of half-edges at each vertex.

Definition 5.3 (Directed cell graph)**.** A **directed cell graph** is a cell graph for which an
arrow is assigned to each edge. An arrow is the same as an ordering of the two half-edges
forming an edge.

Remark 5.4. A directed cell graph is a *quiver*. Since our graph is drawn on an oriented
surface, a directed cell graph carries more information than its underlying quiver structure.
The tail vertex of an arrowed edge is called the *source*, and the head of the arrow the *target*,
in the quiver language.   The quiver structure allows us to deal with non-commutative
Frobenius algebras.

   To label n vertices, we normally use the n-set

$$[n] := \{1, 2, \ldots, n\}.$$

However, it is often easier to use any totally ordered set of n elements for labeling. The main
reason we label the vertices of a cell graph is we wish to assign an element of a K-vector
space A to each vertex. In this article, we consider the case that a cell graph $\gamma \in \Gamma_{g,n}$
defines a linear map

(5.1) $$\Gamma_{g,n} \ni \gamma : A^{\otimes n} \longrightarrow K.$$

The set of values of these functions γ can be more general. We discuss some of the general
cases in [34].
   An effective tool in graph enumeration is edge-contraction operations. Often edge con-
traction leads to an inductive formula for counting problems of graphs. The same edge-
contraction operations acquire algebraic meaning in our consideration.

Definition 5.5 (Edge-contraction operations)**.** There are two types of **edge-contraction
operations** applied to cell graphs.
   • **ECO 1**: Suppose there is a directed edge $\vec{E} = \overrightarrow{p_i p_i}$ in a cell graph $\gamma \in \Gamma_{g,n}$,
   connecting the tail vertex p_i and the head vertex p_j. We *contract* $\vec{E}$ in γ, and put
   the two vertices p_i and p_j together. We use i for the label of this new vertex, and
   call it again p_i. Then we have a new cell graph $\gamma' \in \Gamma_{g,n-1}$ with one less vertices. In
   this process, the topology of the surface on which γ is drawn does not change. Thus
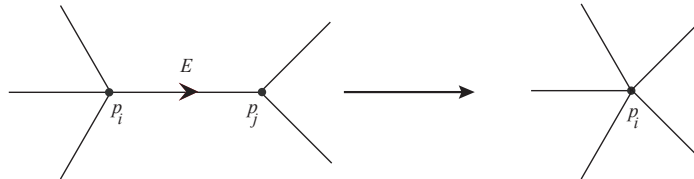   genus g of the graph stays the same.

FIGURE 5.1. Edge-contraction operation ECO 1. The edge bounded by two vertices
p_i and p_j is contracted to a single vertex p_i.

- We use the notation $\vec{E}$ for the edge-contraction operation

(5.2) $$\vec{E} : \Gamma_{g,n} \ni \gamma \longmapsto \gamma' \in \Gamma_{g,n-1}.$$

- **ECO 2**: Suppose there is a directed loop $\vec{L}$ in $\gamma \in \Gamma_{g,n}$ at the i-th vertex p_i. Since a loop in the 1-skeleton of a cell decomposition is a topological cycle on the surface, its contraction inevitably changes the topology of the surface. First we look at the half-edges incident to vertex p_i. Locally around p_i on the surface, the directed loop $\vec{L}$ separates the neighborhood of p_i into two pieces. Accordingly, we put the incident half-edges into two groups. We then break the vertex p_i into two vertices, p_{i_1} and p_{i_2}, so that one group of half-edges are incident to p_{i_1}, and the other group to p_{i_2}. The order of two vertices is determined by placing the loop $\vec{L}$ *upward* near at vertex p_i. Then we name the new vertex on its left by p_{i_1}, and on its right by p_{i_2}.

    Let γ' denote the possibly disconnected graph obtained by contracting $\vec{L}$ and separating the vertex to two distinct vertices labeled by i_1 and i_2.
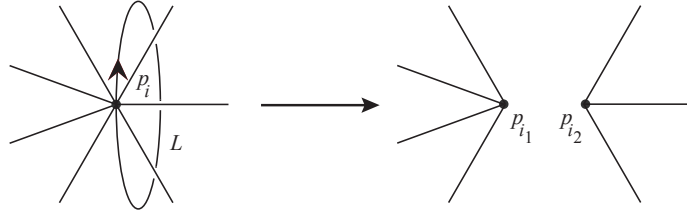

FIGURE 5.2. Edge-contraction operation ECO 2. The contracted edge is a loop $\vec{L}$ of a cell graph. Place the loop so that it is upward near at p_i to which $\vec{L}$ is attached. The vertex p_i is then broken into two vertices, p_{i_1} on the left, and p_{i_2} on the right. Half-edges incident to p_i are separated into two groups, belonging to two sides of the loop near p_i.

- If γ' is connected, then it is in $\Gamma_{g-1,n+1}$. The loop $\vec{L}$ is a *loop of handle*. We use the same notation $\vec{L}$ to indicate the edge-contraction operation

(5.3) $$\vec{L} : \Gamma_{g,n} \ni \gamma \longmapsto \gamma' \in \Gamma_{g-1,n+1}.$$

- If γ' is disconnected, then write $\gamma' = (\gamma_1, \gamma_2) \in \Gamma_{g_1,|I|+1} \times \Gamma_{g_2,|J|+1}$, where

(5.4) $$\begin{cases} g = g_1 + g_2 \\ I \sqcup J = \{1, \ldots, \widehat{i}, \ldots, n\} \end{cases}.$$

    The edge-contraction operation is again denoted by

(5.5) $$\vec{L} : \Gamma_{g,n} \ni \gamma \longmapsto (\gamma_1, \gamma_2) \in \Gamma_{g_1,|I|+1} \times \Gamma_{g_2,|J|+1}.$$

    In this case we call $\vec{L}$ a *separating loop*. Here, vertices labeled by I belong to the connected component of genus g_1, and those labeled by J are on the other component of genus g_2. Let (I_-, i, I_+) (reps. (J_-, i, J_+)) be the reordering of $I \sqcup \{i\}$ (resp. $J \sqcup \{i\}$) in the increasing order. Although we give labeling i_1, i_2 to the two vertices created by breaking p_i, since they belong to distinct graphs, we can simply use i for the label of $p_{i_1} \in \gamma_1$ and the same i for $p_{i_2} \in \gamma_2$. The arrow of $\vec{L}$ translates into the information of ordering among the two vertices p_{i_1} and p_{i_2}.

Remark 5.6. Let us define $m(\gamma) = 2g - 2 + n$ for a graph $\gamma \in \Gamma_{g,n}$. Then every edge-contraction operation reduces $m(\gamma)$ exactly by 1. Indeed, for ECO 1, we have

$$m(\gamma') = 2g - 2 + (n - 1) = m(\gamma) - 1.$$

The ECO 2 applied to a loop of handle produces

$$m(\gamma') = 2(g - 1) - 2 + (n + 1) = m(\gamma) - 1.$$

For a separating loop, we have

$$
+) \quad \begin{array}{c} 2g_1 - 2 + |I| + 1 \\ 2g_2 - 2 + |J| + 1 \\ \hline 2g_1 + 2g_2 - 4 + |I| + |J| + 2 \end{array} \ = \ 2g - 2 + n - 1.
$$

The motivation for our introduction of directed cell graphs is that we need them when we deal with non-commutative Frobenius algebras. The operation of taking disjoint union is symmetric. Therefore, 2D TQFT inevitably leads to a commutative Frobenius algebra. The advantage of our formalism using directed cell graphs is that we can deal with non-commutative Frobenius algebras and non-symmetric tensor products.

For the purpose of presenting the idea of the category of cell graphs as simple as possible, we restrict ourselves to undirected cell graphs in this article. Therefore, we will only recover commutative Frobenius algebras and usual 2D TQFT. A more general theory will be given in [34].

We now introduce the category of cell graphs. The most unusual point we present here is that a morphism between cell graphs is *not* a cell map. Recall that a **cell map** $f : \gamma \longrightarrow \gamma'$ from a cell graph γ to another cell graph γ' is a topological map between 1-dimensional cell complexes. Thus f sends a vertex of γ to a vertex of γ', and an edge of γ to either an edge or a vertex of γ', keeping the incidence relations. In particular, a cell map is continuous with respect to the topological structure on cell graphs indued from the surface on which they are drawn.

Definition 5.7 (Category of cell graphs). The category $\mathcal{CG}$ of cell graphs is defined as follows.

- The set of objects of $\mathcal{CG}$ is the set of all cell graphs:

(5.6) $$Ob(\mathcal{CG}) = \coprod_{g \geq 0, n > 0} \Gamma_{g,n}.$$

- **A morphism**

$$f \in \mathrm{Hom}(\gamma, \gamma')$$

  is a composition of a finite sequence of edge-contraction operations and cell graph automorphisms. In particular, $\mathrm{Hom}(\gamma, \gamma) = \mathrm{Aut}(\gamma)$. If there is no way to bring γ to γ' by consecutive applications of edge-contraction operations and automorphisms, then we define $\mathrm{Hom}(\gamma, \gamma') = \emptyset$, even though there may be cell maps between them.

Remark 5.8. The triple $(\mathcal{CG}, \sqcup, \emptyset)$ forms a symmetric monoidal category.

Remark 5.9. Automorphisms of a cell graph and ECOs of the first kind are cell maps, but ECOs2 operations are not. When an ECO 2 is involved, a morphism between cell graphs does not have to be a cell map. It may not be continuous.

Example 5.10. A few simple examples of morphisms are given below. Note that vertices are all labeled, and automorphisms are required to keep labeling.

(5.7) $$\mathrm{Hom}\,(\bullet, \bullet) = \{id\}.$$

$$(5.8) \qquad\qquad \mathrm{Hom}\left(\bullet, \bullet\!\!-\!\!\!-\!\!\bullet\right) = \emptyset.$$

$$(5.9) \qquad\qquad \mathrm{Hom}(\overset{E_1\ E_2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}, \bullet\!\!-\!\!\!-\!\!\bullet) = \{E_1, E_2\}.$$

$$(5.10) \qquad\qquad \mathrm{Hom}(\overset{E_1\ E_2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}, \bullet) = \{E_1 E_2 = E_2 E_1\}.$$

$$(5.11) \qquad\qquad \mathrm{Hom}\left(\overset{E_1}{\underset{E_2}{\bullet\!\!\bigcirc}}, \bigcirc\right) = \{E_1, E_2 = \sigma(E_1)\}.$$

$$(5.12) \qquad\qquad \mathrm{Hom}\left(\overset{E_1}{\underset{E_2}{\bullet\!\!\bigcirc}}, \bullet\ \ \bullet\right) = \{E_1 E_2 = E_2 E_1\}.$$

In (5.10), we note that $E_1 E_2 := E_1 \circ E_2$ is equal to $E_2 E_1 := E_2 \circ E_1$, because they both produce the same result $\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \to \bullet\!\!-\!\!\!-\!\!\bullet \to \bullet$. The cell graph of the left of (5.11) and (5.12) has an automorphism σ that interchanges E_1 and E_2. Thus as an edge-contraction operation, $E_2 = E_1 \circ \sigma = \sigma(E_1)$. Note that there is a $2:1$ covering cell map for the case of (5.11) that sends both edges E_1 and E_2 on $\overset{E_1}{\underset{E_2}{\bigcirc}}$ to the single loop of $\bigcirc$, and the two vertices on the first graph to the single vertex on the second. Since it is not an edge-contraction, this cell map is not a morphism. The morphism of (5.12) is not a cell map, since it is not continuous.

Let **Vect** be the category of finite-dimensional K-vector spaces. The triple

$$\mathcal{C} = (\mathbf{Vect}, \otimes, K)$$

forms a monoidal category. Again for simplicity, we are concerned only with symmetric tensor products in this article, so we consider $\mathcal{C}$ a *symmetric* monoidal category. A K-object in **Vect** is a pair $(V, \epsilon : V \longrightarrow K)$ consisting of a vector space V and a linear map $\epsilon : V \longrightarrow K$. We denote by $\mathbf{Vect}/K$ the category of K-objects in **Vect**. It has the unique final object $(K, id : K \longrightarrow K)$. Therefore,

$$\mathcal{C}/K = \big(\mathbf{Vect}/K, \otimes, (K, id : K \longrightarrow K)\big)$$

is again a monoidal category. We denote by

$$(5.13) \qquad\qquad \mathcal{F}un(\mathcal{C}/K, \mathcal{C}/K)$$

the **endofunctor category** of the monoidal category $\mathcal{C}/K$, which consists of monoidal functors $\alpha : \mathcal{C}/K \longrightarrow \mathcal{C}/K$ as its objects, and their natural transformations τ as morphisms. Schematically, we have

Here, the triangle on the left shows two objects $(V, f : V \longrightarrow K)$ and $(W, g : W \longrightarrow K)$ of $\mathcal{C}/K$, and a morphism h between them. The prism shape on the right represents two

monoidal endofunctors α and β that assigns

$$V \longmapsto \alpha(V), \qquad V \longmapsto \beta(V)$$
$$W \longmapsto \alpha(W), \qquad W \longmapsto \beta(W),$$

and a natural transformation $\tau : \alpha \longrightarrow \beta$ among them. The final object of $\mathcal{F}un(\mathcal{C}/K, \mathcal{C}/K)$ is the functor

(5.14) $$\phi : (V, f : V \longrightarrow K) \longrightarrow (K, id_K : K \longrightarrow K)$$

which assigns the final object of the codomain $\mathcal{C}/K$ to everything in the domain $\mathcal{C}/K$. With respect to the tensor product and the above functor (5.14) as its identity object, the endofunctor category $\mathcal{F}un(\mathcal{C}/K, \mathcal{C}/K)$ is again a monoidal category.

Definition 5.11 (ECO functor, [34])**.** The ECO functor is a monoidal functor

(5.15) $$\omega : \mathcal{CG} \longrightarrow \mathcal{F}un(\mathcal{C}/K, \mathcal{C}/K)$$

satisfying the following conditions.
   • The graph $\bullet \in \Gamma_{0,1}$ consisting of only one vertex and no edge corresponds to the identity functor

(5.16) $$\omega(\bullet) = id : \mathcal{C}/K \longrightarrow \mathcal{C}/K.$$

   • Each graph $\gamma \in \Gamma_{g,n}$ corresponds to a functor

(5.17) $$\omega(\gamma) : (V, \epsilon : V \longrightarrow K) \longmapsto (V^{\otimes n}, \omega_V(\gamma) : V^{\otimes n} \longrightarrow K).$$

   • Edge-contraction operations correspond to natural transformations.

Let us recall the notion of Frobenius object.

Definition 5.12 (Frobenius object)**.** Let $(\mathcal{C}, \otimes, K)$ be a symmetric monoidal category. A **Frobenius object** is an object $V \in Ob(\mathcal{C})$ together with morphisms

$$m : V \otimes V \longrightarrow V, \quad \mathbf{1} : K \longrightarrow V, \quad \delta : V \longrightarrow V \otimes V, \quad \epsilon : V \longrightarrow K,$$

satisfying the following conditions:
   • $(V, m, \mathbf{1})$ is a monoid object in $\mathcal{C}$.
   • (V, δ, ϵ) is a comonoid object in $\mathcal{C}$.
We also require the compatibility condition (1.8) among morphisms m and δ:

$$
\begin{array}{ccc}
 & V \otimes V \otimes V & \\
{\scriptstyle id \otimes \delta} \nearrow & & \searrow {\scriptstyle m \otimes id} \\
V \otimes V \xrightarrow{\ m\ } V \xrightarrow{\ \delta\ } V \otimes V. \\
{\scriptstyle \delta \otimes id} \searrow & & \nearrow {\scriptstyle id \otimes m} \\
 & V \otimes V \otimes V &
\end{array}
$$

Since we are considering the monoidal category of K-objects in **Vect**, there is a priori no notion of $\mathbf{1}$ in V. The existence of the morphism $\mathbf{1} : K \longrightarrow V$ requires a non-degeneracy condition. The following theorem is proved in [34].

Theorem 5.13 (Generation of Frobenius objects [34])**.** *An object $(V, \epsilon : V \longrightarrow K)$ of $\mathcal{C}/K$ is a Frobenius object if $\omega_V(\bullet\!\!-\!\!\bullet) : V \otimes V \longrightarrow K$ defines a non-degenerate symmetric bilinear form on V.*

6. 2D TQFT FROM CELL GRAPHS

The result of Section 4 tells us that a 2D TQFT can be defined as a system $(A, \omega_{g,n})$ of linear maps

$$(6.1) \qquad \omega_{g,n} : A^{\otimes n} \longrightarrow K$$

defined for all values of $g \geq 0$ and $n \geq 1$, satisfying a set of conditions. The required conditions are the following: First, $(A, \omega_{g,n})$ is a CohFT for $2g - 2 + n > 0$. In addition, we require that

$$(6.2) \qquad \begin{aligned} \omega_{0,1} &= \epsilon : A \longrightarrow K, \\ \omega_{0,2} &= \eta : A \otimes A \longrightarrow K. \end{aligned}$$

In this section we give a different formulation of a 2D TQFT, based on cell graphs and a different set of axioms. Our ultimate goal is to relate 2D TQFT, CohFT, mirror symmetry, topological recursion, and quantum curves. Later in these lectures, we introduce quantum curves. Relations between all these subjects will be discussed elsewhere [34].

Theorem 6.1 (Graph independence [30]). *Let $(A, \epsilon : A \longrightarrow K)$ be a Frobenius object under the ECO functor ω of Definition 5.11. Then every connected cell graph $\gamma \in \Gamma_{g,n}$ gives rise to the same map*

$$(6.3) \qquad \omega_A(\gamma) : A^{\otimes n} \ni v_1 \otimes \cdots \otimes v_n \longmapsto \epsilon(v_1 \cdots v_n \mathbf{e}^g) \in K,$$

where $\mathbf{e}$ is the Euler element of (1.13).

Corollary 6.2 (ECO implies TQFT). *Define $\omega_{g,n}(v_1, \ldots, v_n) = \omega_A(\gamma)(v_1, \ldots, v_n)$ for every $\gamma \in \Gamma_{g,n}$. Then $\{\omega_{g,n}\}$ is a 2D TQFT.*

Proof. Since the value of (6.3) is the same as (4.17), it is a 2D TQFT. $\qquad \square$

The rest of the section is devoted to proving Theorem 6.1. We first give three examples of graph independence.

Lemma 6.3 (Edge-removal lemma). *Let $\gamma \in \Gamma_{g,n}$.*
- ***Case 1.*** *There is a disc-bounding loop L in γ. Let $\gamma' \in \Gamma_{g,n}$ be the graph obtained by simply removing L from γ. Note that we are not contracting L.*
- ***Case 2.*** *The graph γ contains two edges E_1 and E_2 between two distinct vertices p_i and p_j that bound a disc. Let $\gamma' \in \Gamma_{g,n}$ be the graph obtained by removing E_2. Here again, we are just eliminating E_2.*
- ***Case 3.*** *Two loops, L_1 and L_2, in γ are attached to the i-th vertex p_i. If they are homotopic, then let $\gamma' \in \Gamma_{g,n}$ be the graph obtained by removing L_2 from γ.*

In each of the above cases, we have

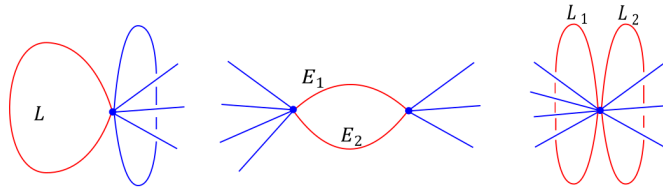$$(6.4) \qquad \omega_A(\gamma)(v_1, \ldots, v_n) = \omega_A(\gamma')(v_1, \ldots, v_n).$$

FIGURE 6.1. Removal of disc-bounding edges.

Proof. Case 1. Contracting a disc-bounding loop attached to p_i creates $(\gamma_0, \gamma') \in \Gamma_{0,1} \times \Gamma_{g,n}$, where γ_0 consists of only one vertex and no edges. The natural transformation corresponding to ECO 1 then gives

$$
\begin{aligned}
\omega_A(\gamma)(v_1, \ldots, v_n) &= \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell) \eta^{ka} \eta^{\ell b} \omega_A(\gamma_0)(e_a) \omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_b, v_{i+1} \ldots, v_n) \\
&= \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell) \eta^{ka} \eta^{\ell b} \eta(\mathbf{1}, e_a) \omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_b, v_{i+1} \ldots, v_n) \\
&= \sum_{b,k,\ell} \eta(v_i, \mathbf{1} \cdot e_\ell) \eta^{\ell b} \omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_b, v_{i+1} \ldots, v_n) \\
&= \sum_{b,\ell} \eta(v_i, e_\ell) \eta^{\ell b} \omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_b, v_{i+1} \ldots, v_n) \\
&= \omega_A(\gamma')(v_1, \ldots, v_{i-1}, v_i, v_{i+1} \ldots, v_n).
\end{aligned}
$$

Case 2. Contracting Edge E_1 makes E_2 a disc-bounding loop at p_i. We can remove it by Case 1. Note that the new vertex is assigned with $v_i v_j$. Restoring E_1 makes the graph exactly the one obtained by removing E_2 from γ. Thus (6.4) holds.

Case 3. Contracting Loop L_1 makes L_2 a disc-bounding loop. Hence we can remove it by Case 1. Then restoring L_1 creates a graph obtained from γ by removing L_2. Thus (6.4) holds. $\qquad\square$

Remark 6.4. The three cases treated above correspond to removing degree 1 and 2 vertices from the dual ribbon graph.

Definition 6.5 (Reduced graph). A cell graph is **reduced** if it does not contain any disc-bounding loops or disc-bounding bigons. In terms of dual ribbon graphs, the dual of a reduced cell graph has no vertices of degree 1 or 2.

We can see from Lemma 6.3, Case 1, that every graph $\gamma \in \Gamma_{0,1}$ gives the same map

(6.5) $$\omega_A(\gamma)(v) = \epsilon(v).$$

Similarly, Cases 2 and 3 of Lemma 6.3 show that every graph $\gamma \in \Gamma_{0,2}$ defines

$$\omega_A(\gamma)(v_1, v_2) = \eta(v_1, v_2).$$

This is because we can remove all edges and loops but one that connects the two vertices. Then by the natural transformation corresponding to ECO 1, the value of the assignment $\omega_A(\gamma)$ is $\epsilon(v_1 v_2) = \eta(v_1, v_2)$.

Proof of Theorem 6.1. We use the induction on $m = 2g - 2 + n$. The base case is $m = -1$, or $(g, n) = (0, 1)$, for which the theorem holds by (6.5). Assume that (6.3) holds for all (g, n) with $2g - 2 + n < m$. Now let $\gamma \in \Gamma_{g,n}$ be a cell graph of type (g, n) such that $2n - 2 + n = m$. Choose an arbitrary straight edge of γ that connects two distinct vertices, say p_i and p_j. Then the natural transformation of contracting this edge to γ' gives

$$\omega_A(\gamma)(v_1, \ldots, v_n) = \omega_A(\gamma')(v_1, \ldots, v_{i-1}, v_i v_j, v_{i+1} \ldots, \widehat{v_j}, \ldots, v_n) = \epsilon(v_1 \ldots v_n \mathbf{e}^g).$$

If we have chosen an arbitrary loop attached to p_i, then its contraction by ECO 2 gives two cases, depending on whether the loop is a loop of handle or a separating loop. For the former case, we have a graph γ', and by appealing to (1.10) and (1.14), we obtain

$$\omega_A(\gamma)(v_1, \ldots, v_n) = \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell) \eta^{ka} \eta^{\ell b} \omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_a, e_b, v_{i+1}, \ldots, v_n)$$

$$= \sum_{a,b,k,\ell} \eta(v_i e_k, e_\ell)\eta^{ka}\eta^{\ell b}\omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_a, e_b, v_{i+1}, \ldots, v_n)$$

$$= \sum_{a,k} \eta^{ka}\omega_A(\gamma')(v_1, \ldots, v_{i-1}, e_a, v_i e_k, v_{i+1}, \ldots, v_n)$$

$$= \sum_{a,k} \eta^{ka}\epsilon(v_1 \cdots v_n \mathbf{e}^{g-1} e_a e_b)$$

$$= \epsilon(v_1 \cdots v_n \mathbf{e}^g).$$

For the case of a separating loop, ECO 2 makes $\gamma \to (\gamma_1, \gamma_2)$, and we have

$$\omega_A(\gamma)(v_1, \ldots, v_n) = \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell)\eta^{ka}\eta^{\ell b}\omega_A(\gamma_1)\big(v_{I_-}, e_a, v_{I_+}\big)\omega_A(\gamma_2)\big(v_{J_-}, e_b, v_{J_+}\big)$$

$$= \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell)\eta^{ka}\eta^{\ell b}\epsilon\left(e_a \prod_{c \in I} v_c \mathbf{e}^{g_1}\right)\epsilon\left(e_b \prod_{d \in J} v_d \mathbf{e}^{g_2}\right)$$

$$= \sum_{a,b,k,\ell} \eta(v_i e_k, e_\ell)\eta^{ka}\eta^{\ell b}\eta\left(\prod_{c \in I} v_c, e_a \mathbf{e}^{g_1}\right)\epsilon\left(e_b \prod_{d \in J} v_d \mathbf{e}^{g_2}\right)$$

$$= \sum_{a,k} \eta^{ka}\eta\left(\prod_{c \in I} v_c \mathbf{e}^{g_1}, e_a\right)\epsilon\left(v_i e_k \prod_{d \in J} v_d \mathbf{e}^{g_2}\right)$$

$$= \epsilon\left(v_i \prod_{c \in I} v_c \mathbf{e}^{g_1} \prod_{d \in J} v_d \mathbf{e}^{g_2}\right)$$

$$= \epsilon(v_1 \cdots v_n \mathbf{e}^{g_1+g_2}).$$

Therefore, no matter how we apply ECO 1 or ECO 2, we always obtain the same result. This completes the proof. □

7. TQFT-valued topological recursion

There is a direct relation between a Frobenius algebra and Gromov-Witten theory when A is given by the big quantum cohomology of a target space. Since these Frobenius algebras are usually infinite-dimensional over the ground field, they do not correspond to a 2D TQFT discussed in the previous sections. But for the case that the target space is 0-dimensional, the TQFT indeed captures the whole Gromov-Witten theory.

In this section, we present a general framework. Fundamental examples are $A = \mathbb{Q}$, which gives ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$, and the center of the group algebra of a finite group $A = Z\mathbb{C}[G]$, which produces Gromov-Witten invariants of the classifying space BG. The first example is considered in [30, 31]. The latter case will be discussed elsewhere [34].

We wish to solve a graph enumeration problem, where as a graph we consider a cell graph, and each of its vertex is *colored* by a *parameter* $v \in A$. We also impose functoriality under the edge-contraction of Definition 5.5 for this coloring. The ECOs reduce the complexity of coloring considerably, because the functoriality makes the coloring process graph independent, as we have shown in the last section. Thus the answer is just the number of graphs times the value $\epsilon(v_1 \cdots v_n \mathbf{e}^g)$ for each topological type (g, n). Here comes the difficulty: there are infinitely many graphs for each topological type, since we allow multiple edges and loops. The standard idea for such counting problem is to appeal to the *Laplace transform*, which is introduced by Laplace for this particular context of counting an infinite

number of objects. Let us denote by

$$\Gamma_{g,n}(\mu_1, \ldots, \mu_n) \tag{7.1}$$

the set of all cell graphs with labeled vertices of degrees $(\mu_1, \ldots, \mu_n)$. Denoting by c_d the number of d-cells in a cell-decomposition of a surface of genus g, $d = 0, 1, 2$, we have $n = c_0$, $\sum_{i=1}^{n} \mu_i = 2c_1$, and $2 - 2g = n - c_1 + c_2$. Therefore, (7.1) is a finite set.

Counting processes become easier if we do not have any object with non-trivial automorphism. There are many ways to eliminate automorphism, for example, the minimalistic way of imposing the least possible conditions, or an excessive way to kill automorphisms but for the most objects the conditions are redundant. Actually, it is known that most of the cell graphs counted in (7.1) are without any non-trivial automorphisms. Since any possible automorphism induces a cyclic permutation of half-edges incident to each vertex, the easiest way to disallow any automorphism is to assign an outgoing arrow to *one* of these half-edges, as in [35, 85] (but not as a quiver). An automorphism should preserve the arrowed half-edges, in addition to labeled vertices. We denote by

$$\widehat{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n) \tag{7.2}$$

the set of **arrowed cell graphs** with labeled vertices of degrees $(\mu_1, \ldots, \mu_n)$, and by

$$C_{g,n}(\mu_1, \ldots, \mu_n) := \left| \widehat{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n) \right| \tag{7.3}$$

its cardinality. This number is always a non-negative integer, and $C_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m}$ is the m-th Catalan number.

We use the notation $\omega_A(\gamma) = \omega_{g,n} : A^{\otimes n} \longrightarrow K$ of Corollary 6.2 for $\gamma \in \widehat{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n)$. We are interested in considering the $C_{g,n}(\mu_1, \ldots, \mu_n)$-weighted TQFT

$$C_{g,n}(\mu_1, \ldots, \mu_n) \cdot \omega_{g,n} : A^{\otimes n} \longrightarrow K, \tag{7.4}$$

and applying the edge-contraction operations to these maps. We contract the edge of γ that carries the outgoing arrow at the first vertex p_1, then place a new arrow to the half-edge next to the original half-edge with respect to the cyclic ordering induced by the orientation of the surface. If the edge that carries the outgoing arrow at p_1 is a loop, then after splitting p_1, we place another arrow to the next half-edge at each of the two newly created vertices, again next to the original loop. From this process we obtain the following counting formula:

Proposition 7.1. *The 2D TQFT weighted by the number of arrowed cell graphs satisfies the following equation.*

$$
\begin{aligned}
&C_{g,n}(\mu_1, \ldots, \mu_n) \cdot \omega_{g,n}(v_1, \ldots, v_n) \\
&= \sum_{j=2}^{n} \mu_j C_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_n) \cdot \omega_{g,n-1}(v_1 v_j, v_2, \ldots, \widehat{v_j}, \ldots, v_n) \\
&\quad + \sum_{\alpha+\beta=\mu_1-2} C_{g-1,n+1}(\alpha, \beta, \mu_2, \ldots, \mu_n) \cdot \omega_{g-1,n+1}\big(\delta(v_1), v_2, \ldots, v_n\big) \\
&\quad + \sum_{\substack{\alpha+\beta=\mu_1-2}} \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\}}} \sum_{a,b,k,\ell} \eta(v_1, e_k e_\ell) \eta^{ka} \eta^{\ell b} \\
&\qquad \times \big(C_{g_1,|I|+1}(\alpha, \mu_I) \cdot \omega_{g_1,|I|+1}(e_a, v_I)\big) \big(C_{g_2,|J|+1}(\beta, \mu_J) \cdot \omega_{g_2,|J|+1}(e_b, v_J)\big).
\end{aligned}
\tag{7.5}
$$

This is exactly the same formula of [35, 85, 102] multiplied by

$$\omega_{g,n}(v_1, \ldots, v_n) = \epsilon(v_1 \ldots v_n \mathbf{e}^g).$$

Let us now consider a **Frobenius algebra twisted topological recursion**. To simplify the notation, we adopt the following way of writing:

$$(7.6) \qquad (f \otimes h) \circ \delta = \sum_{a,b,k,\ell} \eta(\bullet, e_k e_\ell) \eta^{ka} \eta^{\ell b} f(e_a) h(e_b),$$

where $f, h : A \longrightarrow K$ are linear functions on A.

First let us review the original topological recursion of [39] defined on a **spectral curve** Σ, which is just a disjoint union of r copies of open discs. Let U be the unit disc centered at 0 of the complex z-line. We choose r sets of functions (x_α, y_α), $\alpha = 1, \ldots, r$, defined on U with Taylor expansions

$$(7.7) \qquad x_\alpha(z) = z^2 + \sum_{k=3}^\infty a_{\alpha,k} z^k, \qquad y_\alpha(z) = z + \sum_{k=2}^\infty b_{\alpha,k} z^k,$$

and a meromorphic 1-form (Cauchy kernel)

$$(7.8) \qquad \omega_\alpha^{p-q}(z) = \frac{dz}{z-p} - \frac{dz}{z-q} + \omega_\alpha$$

on U, where $p, q \in U$, and ω_α is a holomorphic 1-form on U. Since each $x_\alpha : U \longrightarrow \mathbb{C}$ is a $2:1$ map, we have an involution on U that keeps the same x_α-value:

$$(7.9) \qquad \sigma_\alpha : U \longrightarrow U, \qquad x_\alpha\big(\sigma_\alpha(z)\big) = x_\alpha(z).$$

To avoid confusion, we label r copies of U by $U_1, \ldots, U_r$, and consider the functions (x_α, y_α) to be defined on U_α. The **topological recursion** is the following recursive equation

$$(7.10) \quad W_{g,n}(z_1, \ldots, z_n) = \frac{1}{2\pi i} \sum_{\alpha=1}^r \oint_{\partial U_\alpha} \frac{\omega_\alpha^{\sigma_\alpha(z)-z}(z_1)}{\big(y_\alpha(\sigma_\alpha(z)) - y_\alpha(z)\big) dx_\alpha(z)}$$

$$\times \left[W_{g-1,n+1}\big(z, \sigma_\alpha(z), z_2, \ldots, z_n\big) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}}^{\text{No } (0,1)} W_{g_1,|I|+1}(z, z_I) W_{g_2,|J|+1}\big(\sigma_\alpha(z), z_J\big) \right]$$

on symmetric meromorphic n-differential forms $W_{g,n}$ defined on the disjoint union $U_1^n \sqcup \cdots \sqcup U_r^n$ for $2g - 2 + n > 0$. Here, $z_I = (z_i)_{i \in I}$, and "No $(0,1)$" in the summation means the partition $g = g_1 + g_2$ and the set partition $I \sqcup J = \{2, \ldots, n\}$ do not allow $g_1 = 0$ and $I = \emptyset$, or $g_2 = 0$ and $J = \emptyset$. The integration is performed with respect to $z \in \partial U$. Note that the differential form in the big bracket [ ] in (7.10) is a symmetric quadratic differential in the variable $z \in U_\alpha$. The expression $1/dx_\alpha(z)$ is a meromorphic section of $K_{U_\alpha}^{-1}$, where K_{U_α} denotes the canonical sheaf of holomorphic 1-forms on U_α. Thus the integrand of the recursion becomes a meromorphic 1-form on U_α in the z-variable, for which the integration is performed. The multiplication by $\omega_\alpha^{\sigma_\alpha(z)-z}(z_1)$ is simply the symmetric tensor product with a 1-form proportional to dz_1. The 1-form $W_{0,1}$ and the 2-form $W_{0,2}$ are defined separately:

$$(7.11) \qquad W_{0,1}(z) := \sum_{\alpha=1}^r y_\alpha(z) dx_\alpha(z)$$

is defined on the disjoint union $U_1 \sqcup \cdots \sqcup U_r$. If $z \in U_\alpha$, then $W_{0,1}(z) = y_\alpha(z) dx_\alpha(z)$. In the form of (7.10), however, $W_{0,1}$ does not appear anywhere. Similarly, $W_{0,2}$ is defined by

$$(7.12) \qquad W_{0,2}(z_1, z_2) := d_{z_1} \omega_\alpha^{z_1-q}(z_2)$$

if $z_1, z_2 \in U_\alpha$. Here, the constant $q \in U_\alpha$ does not play any role. This 2-form explicitly appears in the recursion part (the terms in the big bracket [ ]) of the formula.

Definition 7.2. A **Frobenius algebra twisted topological recursion** for
$$\mathcal{W}_{g,n} = \mathcal{W}_{g,n}(z_1, \ldots, z_n) : A^{\otimes n} \longrightarrow K$$

is the following formula:

(7.13)
$$\mathcal{W}_{g,n}(z_1, \ldots, z_n; v_1, \ldots, v_n) = \frac{1}{2\pi i} \sum_{\alpha=1}^{r} \oint_{\partial U_\alpha} \sum_{a,b,k,\ell} \mathcal{K}_\alpha\big(z, \sigma_\alpha(z), z_1; e_k, e_\ell, v_1\big)$$

$$\times \left[\mathcal{W}_{g-1,n+1}\big(z, \sigma_\alpha(z), z_2, \ldots, z_n; e_a, e_b, v_2, \ldots, v_n\big) \right.$$

$$\left. + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\}}}^{\text{No } (0,1)} \mathcal{W}_{g_1,|I|+1}\big(z, z_I; e_a, v_I\big) \mathcal{W}_{g_2,|J|+1}\big(\sigma_\alpha(z), z_J; e_b, v_J\big) \right].$$

Here

(7.14)
$$\mathcal{K}_\alpha\big(z, \sigma_\alpha(z), z_1; e_k, e_\ell, v_1\big) = \frac{\omega_\alpha^{\sigma_\alpha(z)-z}(z_1)\eta(e_k e_\ell, v_1)\eta^{ka}\eta^{\ell b}}{\big(y_\alpha(\sigma_\alpha(z)) - y_\alpha(z)\big) dx_\alpha(z)}$$

is the integration-summation kernel. Symbolically we can write (7.13) as

(7.15)
$$\mathcal{W}_{g,n} = \frac{1}{2\pi i} \sum_{\alpha=1}^{r} \oint_{\partial U_\alpha} K_\alpha\big(z, \sigma_\alpha(z), z_1\big)$$

$$\times \left[\mathcal{W}_{g-1,n+1} \circ \delta + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\}}}^{\text{No } (0,1)} \big(\mathcal{W}_{g_1,|I|+1} \otimes \mathcal{W}_{g_2,|J|+1}\big) \circ \delta \right]$$

with the usual integration kernel

$$K_\alpha\big(z, \sigma_\alpha(z), z_1\big) := \frac{\omega_\alpha^{\sigma_\alpha(z)-z}(z_1)}{\big(y_\alpha(\sigma_\alpha(z)) - y_\alpha(z)\big) dx_\alpha(z)}.$$

Theorem 7.3. *The topological recursion (7.13) uniquely determines the $(A^{\otimes n})^*$-valued n-linear differential form $\mathcal{W}_{g,n}$ from the initial data $\mathcal{W}_{0,2}$. If the initial data is given by*
$$\mathcal{W}_{0,2}(z_1, z_2; v_1, v_2) = W_{0,2}(z_1, z_2) \cdot \eta(v_1, v_2)$$

for a 2-form (7.12), then there exists a solution $\{W_{g,n}\}$ of the topological recursion (7.10) and a 2D TQFT $\{\omega_{g,n}\}$ such that

(7.16)
$$\mathcal{W}_{g,n}(z_1, \ldots, z_n; v_1, \ldots, v_n) = W_{g,n}(z_1, \ldots, z_n) \cdot \omega_{g,n}(v_1, \ldots, v_n).$$

Proof. The proof is done by induction on $m = 2g - 2 + n$ with the base case $m = 0$. We assume that (7.16) holds for all (g, n) such that $2g - 2 + n < m$, and use the value $\omega_{g,n} = \epsilon(v_1 \cdots v_n \mathbf{e}^g)$ given by (4.17) for the values of (g, n) in the range of induction hypothesis. Then by (7.13) and functoriality under ECO 2, we conclude that (7.16) also holds for all (g, n) such that $2g - 2 + n = m$. □

Remark 7.4. In comparison to edge-contraction operations, we note that the multiplication case ECO 1 does not seem to have a counterpart in an explicit way. It is actually included in the terms involving $g_1 = 0, |I| = 1$ and $g_2 = 0, |J| = 1$ in the partition sum, and (1.15) is used to change the comultiplication to multiplication. More precisely, if $I = \{i\}$, then it gives a term

$$
\mathcal{W}_{0,2}(z, z_i; e_a, v_i)\mathcal{W}_{g,n-1}\big(\sigma_\alpha(z), z_2, \ldots, \widehat{z_i}, \ldots, z_n; e_b, v_2, \ldots, \widehat{v_i}, \ldots, v_n\big)
$$
$$
= W_{0,2}(z, z_i)W_{g,n-1}\big(\sigma_\alpha(z), z_2, \ldots, \widehat{z_i}, \ldots, z_n\big)
$$
$$
\cdot\, \omega_{0,2}(e_a, v_i)\omega_{g,n-1}(e_b, v_2, \ldots, \widehat{v_i}, \ldots, v_n)
$$

in the partition sum, assuming the induction hypothesis. We also note that

$$
\sum_{a,b,k,\ell} \eta(e_k e_\ell, v_1)\eta^{ka}\eta^{\ell b}\omega_{0,2}(e_a, v_i)\omega_{g,n-1}(e_b, v_2, \ldots, \widehat{v_i}, \ldots, v_n)
$$
$$
= \sum_{b,\ell} \eta^{\ell b}\omega_{0,2}(v_1 e_\ell, v_i)\omega_{g,n-1}(e_b, v_2, \ldots, \widehat{v_i}, \ldots, v_n)
$$
$$
= \sum_{b,\ell} \eta^{\ell b}\eta(v_1 v_i, e_\ell)\omega_{g,n-1}(e_b, v_2, \ldots, \widehat{v_i}, \ldots, v_n)
$$
$$
= \omega_{g,n-1}(v_1 v_i, v_1, \ldots, \widehat{v_i}, \ldots, v_n).
$$

By taking the Laplace transform of (7.5) using the method of [35, 85], we obtain a solution $\mathcal{W}_{g,n} = W^D_{g,n}(z_1, \ldots, z_n) \cdot \omega_{g,n}$ to the topological recursion, where $W^D_{g,n}$ is given by [35, (4.14)] with respect to a global spectral curve of [35, Theorem 4.3], and $\omega_{g,n}$ is the TQFT corresponding to the Frobenius algebra A.

Part 2. **Quantization of Higgs Bundles**

8. Quantum curves

The cohomology ring $H^*(X, \mathbb{C})$ of a compact Kähler variety X is a $\mathbb{Z}/2\mathbb{Z}$-graded, $\mathbb{Z}/2\mathbb{Z}$-commutative Frobenius algebra over $\mathbb{C}$. The genus 0 Gromov-Witten invariants of X define the *big quantum cohomology* of X, which is a quantum deformation of the cohomology ring. If the mirror symmetry is established for X, then the information of big quantum cohomology of X is supposed to be encoded in holomorphic geometry of a mirror dual space Y. Gromov-Witten invariants are generalized to all values of (g, n) with $2g - 2 + n > 0, g \geq 0, n > 0$. The question is:

Question 8.1. *What should be the holomorphic geometry on Y that captures all genera Gromov-Witten invariants of X through mirror symmetry?*

Since the transition from $g = 0$ to all values of $g \geq 0$ is indeed a *quantization*, the holomorphic object on Y that should capture higher genera Gromov-Witten invariants of X is a *quantum geometry* of Y. A naïve guess may be that it should be a $\mathcal{D}$-module that represents Y as its classical limit. Hence the $\mathcal{D}$-module is not defined on Y. Then where does it live?

The idea of **quantum curves** concerns a rather restricted situation, when the mirror geometry Y is captured by an algebraic, or an analytic, curve. Typical examples are the mirror of toric Calabi-Yau orbifolds of three dimensions. Geometry of the mirror Y of a toric Calabi-Yau 3-fold is encoded in a complex curve known as the *mirror curve*. Another situation is enumeration problems of various Hurwitz-type coverings of $\mathbb{P}^1$, and also many

decorated graphs on surfaces. For these examples, although there are no "spaces" X, the mirror geometry exists, and is indeed a curve. These mirror curves are special cases of more general notion of **spectral curves**. Besides mirror symmetry, spectral curves appear in theory of integrable systems, random matrix theory, topological recursion, and Hitchin theory of Higgs bundles. A spectral curve Σ has two common features. The first one is that it is a Lagrangian subvariety of a holomorphic symplectic surface. The other is the existence of a projection $\pi : \Sigma \longrightarrow C$ to another curve C, called a *base curve*. The quantum curve is a $\mathcal{D}$-module on the base curve C such that its *semi-classical limit* is the spectral curve realized in the cotangent bundle $\Sigma \subset T^*C$.

From the analogy of 2D TQFT and CohFT, we note that a spectral curve is already a quantized object, since it corresponds to quantum cohomology. The even part $H^{even}(X, \mathbb{C})$, which is a commutative Frobenius algebra, is not the one that corresponds to a spectral curve. In this sense, CohFT is a result of two quantizations: the first one from classical cohomology to a big quantum cohomology through n-point Gromov-Witten invariants of genus 0; and the second quantization is the passage from genus 0 to all genera.

Remark 8.2. We note that quantum cohomology itself is a Frobenius algebra, though it is not finite dimensional, because it requires the introduction of Novikov ring. The even degree part forms a commutative Frobenius algebra, yet it does not correspond to a 2D TQFT in the way we presented in Part 1.

Now let us turn to the topic of Part 2. The prototype of quantization is a Schrödinger equation. Consider a harmonic oscillator of mass 1, energy E, and the spring constant $1/4$ in one dimension. It has a geometric description as an elliptical motion of a constant angular momentum in the phase space, or the cotangent bundle of the real axis. Here, the spectral curve is an ellipse

$$\frac{1}{4}x^2 + y^2 = E$$

in a real symplectic plane. The quantization of this spectral curve is the quantization of the harmonic oscillator, which is a second order stationary Schrödinger equation in one variable:

$$(8.1) \qquad \left(-\hbar^2 \frac{d^2}{dx^2} + \frac{1}{4}x^2 - E\right)\psi(x, \hbar) = 0.$$

The quantization we discuss in Part 2 is in complete parallelism to quantization of harmonic oscillator. As a holomorphic symplectic surface, we use $(\mathbb{C}^2, dx \wedge dy)$. A plane quadric

$$(8.2) \qquad \frac{1}{4}x^2 - y^2 = 1$$

is an example of a spectral curve. In complex coordinates, we identify $y = d/dx$, ignoring the imaginary unit. An example of quantization of (8.2) is a Schrödinger equation

$$(8.3) \qquad \left(\left(\hbar \frac{d}{dx}\right)^2 + 1 - \frac{1}{2}\hbar - \frac{1}{4}x^2\right)\psi(x, \hbar) = 0,$$

which is essentially the same as quantum harmonic oscillator equation (8.1), and is known as the Hermite-Weber equation. Its solutions are all well studied.

Question 8.3. *Why do we care this well-known classical differential equation?*

A surprising answer [29, 31, 35, 85] to this question is that we find the intersection numbers of $\overline{\mathcal{M}}_{g,n}$ through the asymptotic expansion! First we apply a gauge transformation

$$
(8.4)
\begin{aligned}
& e^{-\frac{1}{4\hbar}x^2}\left(\left(\hbar\frac{d}{dx}\right)^2 + 1 - \frac{1}{2}\hbar - \frac{1}{4}x^2\right)e^{\frac{1}{4\hbar}x^2}Z(x,\hbar) \\
& = \left(\hbar\frac{d^2}{dx^2} + \hbar x\frac{d}{dx} + 1\right)Z(x,\hbar) = 0,
\end{aligned}
$$

where $Z(x,\hbar) = e^{-\frac{1}{4\hbar}x^2}\psi(x,\hbar)$. Recall the integer valued function $C_{g,n}(\mu_1,\ldots,\mu_n)$ of (7.3), and define their generating functions by

$$
(8.5) \qquad F_{g,n}(x_1,\ldots,x_n) := \sum_{\mu_1,\ldots,\mu_n>0}\frac{C_{g,n}(\mu_1,\ldots,\mu_n)}{\mu_1\cdots\mu_n}x_1^{-\mu_1}\cdots x_n^{-\mu_n}.
$$

It is discovered in [35] that the derivatives $W_{g,n} = d_1\cdots d_nF_{g,n}$ satisfy the topological recursion based on the spectral curve (0.1), which is the semi-classical limit of (8.4). With an appropriate adjustment for $(g,n) = (0,1)$ and $(0,2)$, we have the following all-order WKB expansion formula [31, 35, 85]:

$$
(8.6) \qquad Z(x,\hbar) = \exp\left(\sum_{g,n}\frac{1}{n!}\hbar^{2g-2+n}F_{g,n}(x,\ldots,x)\right).
$$

We find ([35]) that if we change the coordinate from x to t by

$$
(8.7) \qquad x = x(t) = \frac{t+1}{t-1} + \frac{t-1}{t+1},
$$

then $F_{g,n}\big(x(t_1), x(t_2),\ldots,x(t_n)\big)$ is a Laurent polynomial for each (g,n) with $2g-2+n>0$. The coordinate change (8.7) is identified in [29] as a **normalization** of the singular curve (0.1) in the Hirzebruch surface $\mathbb{P}\big(K_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}\big)$ by a sequence of blow-ups. The highest degree part of this Laurent polynomial is a homogeneous polynomial of degree $6g-6+3n$

$$
(8.8) \quad F_{g,n}^{\text{highest}}(t_1,\ldots,t_n) = \frac{(-1)^n}{2^{2g-2+n}}\sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}\prod_{i=1}^n\left(|2d_i-1|!!\left(\frac{t_i}{2}\right)^{2d_i+1}\right),
$$

where the coefficients

$$
(8.9) \qquad \langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}
$$

are cotangent class intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$. Topological recursion is a mechanism to calculate all $F_{g,n}(x_1,\ldots,x_n)$ from the single equation (8.3), or equivalently, (8.4). Thus the quantum curve (8.3) has the information of all intersection numbers (8.9). These are the topics discussed in our previous lectures [31].

Although the following topic is not what we deal with in this article, for the moment let us consider a symplectic surface $(\mathbb{C}^*\times\mathbb{C}^*, d\log x\wedge d\log y)$. As a spectral curve, we use the zero locus of the **A-polynomial** $A_K(x,y)$ of a knot defined in [14]. For a given knot $K\subset S^3$, the $SL_2(\mathbb{C})$-character variety

$$
(8.10) \qquad \text{Hom}\big(\pi_1(S^3\setminus K), SL_2(\mathbb{C})\big)/\!\!/SL_2(\mathbb{C})
$$

of the fundamental group of the knot complement determines an algebraic curve in $(\mathbb{C}^*)^2$ defined by $A_K(x,y)\in\mathbb{Z}[x,y]$. Here, $(\mathbb{C}^*)^2$, or to be more precise, $(\mathbb{C}^*)^2/(\mathbb{Z}/2\mathbb{Z})$, is the

$SL_2(\mathbb{C})$-character variety of the fundamental group of the torus $T^2 = S^1 \times S^1$, which is the boundary of the knot complement.

Now consider a function $f(q,n)$ in 2 variables $(q,n) \in \mathbb{C}^* \times \mathbb{Z}_+$, and define operators

(8.11)
$$\begin{cases} (\widehat{x}f)(q,n) := q^n f(q,n) \\ (\widehat{y}f)(q,n) := f(q, n+1), \end{cases}$$

following [42, 45]. These operators satisfy the commutation relation

$$\widehat{x} \cdot \widehat{y} = q\widehat{y} \cdot \widehat{x}.$$

The procedure of changing $x \longmapsto \widehat{x}$ and $y \longmapsto \widehat{y}$ is the **Weyl quantization**. Garoufalidis [45] conjectures that there exists a *quantization* of the A-polynomial such that

(8.12)
$$\widehat{A}_K(\widehat{x}, \widehat{y}; q) J_K(q,n) = 0,$$

where $J_K(q,n)$ is the colored Jones polynomial of the knot K indexed by the dimension n of the irreducible representation of $SL_2(\mathbb{C})$. Here, the quantization means that the operator $\widehat{A}_K(\widehat{x}, \widehat{y}; q)$ recovers the A-polynomial by the restriction

$$\widehat{A}_K(x, y; 1) = A_K(x, y).$$

This relations is the semi-classical limit, which provides the initial condition of the WKB analysis.

A geometric definition of a quantum curve that arises as the quantization of a Hitchin spectral curve is developed in [32], based on the work of [27] that solves a conjecture of Gaiotto [43, 44]. The WKB analysis of the quantum curve [28, 29, 31, 60, 61] is performed by applying the topological recursion of [39]. The Hermite-Weber differential equation is an example of this geometric theory. Although there have been many speculations [52] of the applicability of the topological recursion to low-dimensional topology, still there is no counterpart of the Hitchin type geometric theory for the case of the quantization of A-polynomials. The appearance of the *modularity* in this context [23, 63, 106] is a tantalizing phenomenon, on which there has been a great advancement.

In the following sections, we unfold a different story of quantum curves. In geometry, there is a process parallel to the passage from a spectral curve (0.1) to a quantum curve (8.4). This process is a *journey* from the moduli space of Hitchin spectral curves to the moduli space of opers [26, 32]. The quantization parameter, the Planck constant $\hbar$ of (8.4), acquires a geometric meaning in this process. We begin the story with finding a coordinate independent description of global differential equations of order 2 on a compact Riemann surface.

9. PROJECTIVE STRUCTURES, OPERS, AND HIGGS BUNDLES

In [28], the authors have given a definition of partial differential equation version of topological recursion for Hitchin spectral curves. When the spectral curve is a double sheeted covering of the base curve, we have shown that this PDE topological recursion produces a quantum curve of the Hitchin spectral curve through WKB analysis. The mechanism is explained in detail in [29, 31].

WKB analysis is certainly one way to describe quantization. Yet the passage from spectral curves to their quantization is purely geometric. This point of view is adopted in [32], based on our work [27] on a conjecture of Gaiotto [43]. Our statement is that the quantization process is a biholomorphic map from the moduli space of Hitchin spectral curves to the moduli space of **opers** [6]. In this section, we introduce the notion of opers, and

construct the biholomorphic map mentioned above. In this passage, we give a geometric interpretation of the Planck *constant* $\hbar$ (9.29) as a deformation parameter of vector bundles and connections. For simplicity of presentation, we restrict our attention to $SL_2(\mathbb{C})$-opers. We refer to [27, 32] for more general cases.

To deal with linear differential equation of order higher then or equal to 2 globally on a compact Riemann surface C, we need a *projective coordinate system.* If ω is a global holomorphic or meromorphic 1-form on C, then

$$(9.1) \qquad (d + \omega)f = 0$$

is a first order linear differential equation that makes sense globally on C. This is because $d + \omega$ is a homomorphism from $\mathcal{O}_C$ to K_C, allowing singularities if necessary. Here, K_C is the sheaf of holomorphic 1-forms on C. Of course existence of a non-trivial global solution of (9.1) is a different matter, because it is equivalent to $\omega = -d\log f$.

Suppose we have a second order differential equation

$$(9.2) \qquad \left(\frac{d^2}{dz^2} - q(z)\right) f(z) = 0$$

locally on C with a local coordinate z. Clearly $\frac{d^2}{dz^2}$ is not globally defined, in contrast to the exterior differentiation d in (9.1). What is the requirement for (9.2) to make sense coordinate free, then? Let us find an answer by imposing

$$(9.3) \qquad dz^2 \left[\left(\frac{d}{dz}\right)^2 - q(z)\right] f(z) = 0 \iff dw^2 \left[\left(\frac{d}{dw}\right)^2 - q(w)\right] f(w) = 0.$$

We wish the shape of the equation to be analogous to (9.1). Since ω is a 1-form, we impose that $q \in H^0(C, K_C^{\otimes 2})$ is a global quadratic differential on C. Under a coordinate change $w = w(z)$, q satisfies $q(z)dz^2 = q(w)dw^2$.

How should we think about a solution f? For (9.2) to have a coordinate free meeting, we need to identify the line bundle L on C to which f belongs. Let us denote by $e^{-g(w(z))}$ the transition function of L with respect to the coordinate patch $w = w(z)$. A function $g(w) = g(w(z))$ will be determined later. A solution f satisfies the coordinate condition

$$(9.4) \qquad e^{-g(w(z))} f(w(z)) = f(z).$$

Then

$$0 = dz^2 \cdot e^{g(w(z))} \left[\left(\frac{d}{dz}\right)^2 - q(z)\right] f(z)$$

$$= dz^2 \cdot e^{g(w(z))} \left[\left(\frac{d}{dz}\right)^2 - q(z)\right] e^{-g(w(z))} f(w(z))$$

$$= dz^2 \cdot \left(e^{g(w(z))} \frac{d}{dz} e^{-g(w(z))}\right)^2 f(w(z)) - dw^2 q(w) f(w)$$

$$= dz^2 \cdot \left(\frac{d}{dz} - g_w(w)w'\right)^2 f(w(z)) - dw^2 q(w) f(w)$$

$$= dz^2 \cdot \left[\left(\frac{d}{dz}\right)^2 - 2g_w(w)w'\frac{d}{dz} - \left((g_w(w)w')' - (g_w(w)w')^2\right)\right] f(w(z)) - dw^2 q(w) f(w)$$

$$= dz^2 \cdot \left[(f_w(w)w')' - 2g_w(w)w' f_w(w)w' - \left((g_w(w)w')' - (g_w(w)w')^2\right) f(w)\right]$$

$$- dw^2 q(w) f(w)$$

$$= f_{ww}(w) \left(\frac{dw}{dz}\right)^2 dz^2 + f_w(w) w'' dz^2 - 2g_w(w) f_w(w) dw^2$$

$$- \left((g_w(w)w')' - (g_w(w)w')^2\right) f(w) dz^2 - dw^2 q(w) f(w)$$

$$= dw^2 \cdot \left[\left(\frac{d}{dw}\right)^2 - q(w)\right] f(w) + f_w(w) \left(w'' - 2g_w(w)(w')^2\right) dz^2$$

$$- \left((g_w(w)w')' - (g_w(w)w')^2\right) f(w) dz^2,$$

where $w' = dw/dz$. Therefore, for (9.3) to hold, we need

(9.5) $$w'' - 2g_w(w)(w')^2 = 0$$

(9.6) $$\left(g_w(w)w'\right)' - \left(g_w(w)w'\right)^2 = 0.$$

From (9.5) we find

(9.7) $$g_w(w)w' = \frac{1}{2}\frac{w''}{w'}.$$

Substitution of (9.7) in (9.6) yields

(9.8) $$s_z(w) := \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = 0.$$

We thus encounter the **Schwarzian derivative** $s_z(w)$. Since (9.7) is equivalent to

$$g(w)' = \frac{1}{2}\big(\log(w')\big)',$$

we obtain $g(w) = \frac{1}{2}\log w'$. Here, the constant of integration is 0 because $g(z) = 0$ when $w = z$. Then (9.4) becomes

$$f(z) = e^{-\frac{1}{2}\log w'} f(w) \iff f(z) = \sqrt{\frac{dz}{dw}} f(w),$$

which identifies the line bundle to which f belongs: $f(z) \in K_C^{-\frac{1}{2}}$. We conclude that the coordinate change $w = w(z)$ should satisfy the vanishing of the Schwarzian derivative $s_z(w) \equiv 0$, and the solution $f(z)$ should be considered as a (multivalued) section of the inverse half-canonical $K_C^{-\frac{1}{2}}$. The vanishing of the Schwarzian derivative dictates us to use a complex projective coordinate system of C.

A holomorphic connection in a vector bundle E on C is a $\mathbb{C}$-linear map $\nabla : E \longrightarrow K_C \otimes E$ satisfying the Leibniz condition $\nabla(fs) = f\nabla(s) + df \otimes s$ for $f \in \mathcal{O}_C$ and $s \in E$. Since C is a complex curve, every connection on C is automatically flat. Therefore, ∇ gives rise to a holonomy representation

(9.9) $$\rho : \pi_1(C) \longrightarrow G$$

of the fundamental group $\pi_1(C)$ of the curve C into the structure group G of the vector bundle E. A flat connection ∇ is **irreducible** if the image of the holonomy representation (9.9) is Zariski dense in the complex algebraic group G. In our case, since $G = SL_2(\mathbb{C})$, this requirement is equivalent to that $Im(\rho)$ contains two non-commuting elements of G. The moduli space $\mathcal{M}_{\mathrm{deR}}$ of irreducible holomorphic connections (E, ∇) in a G-bundle E has been constructed (see [100]).

Definition 9.1 ($SL_2(\mathbb{C})$-opers)**.** Consider a point $(E, \nabla) \in \mathcal{M}_{\mathrm{deR}}$ consisting of an irreducible holomorphic $SL_2(\mathbb{C})$-connection $\nabla : E \longrightarrow E \otimes K_C$ acting on a vector bundle E. It is an $SL_2(\mathbb{C})$**-oper** if there is a line subbundle $F \subset E$ such that the connection induces an $\mathcal{O}_C$-module isomorphism

$$(9.10) \qquad \overline{\nabla} : F \xrightarrow{\sim} (E/F) \otimes K_C.$$

The notion of oper is a generalization of projective structures on a Riemann surface. Every compact Riemann surface C admits a projective structure subordinating the given complex structure (see [53]). For our purpose of quantization of Hitchin spectral curves associated with *holomorphic* Higgs bundles, let us assume that $g(C) \geq 2$ in what follows. When we allow singularities, we can relax this condition and deal with genus 0 and 1 cases. A *complex projective coordinate system* is a coordinate neighborhood covering

$$C = \bigcup_\alpha U_\alpha$$

with a local coordinate z_α of U_α such that for every $U_\alpha \cap U_\beta$, we have a fractional linear transformation

$$(9.11) \qquad z_\alpha = \frac{a_{\alpha\beta} z_\beta + b_{\alpha\beta}}{c_{\alpha\beta} z_\beta + d_{\alpha\beta}}, \qquad f_{\alpha\beta} := \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \in SL_2(\mathbb{C}),$$

satisfying a cocycle condition $[f_{\alpha\beta}][f_{\beta\gamma}] = [f_{\alpha\gamma}]$. Here, $[f_{\alpha\beta}]$ is the class of $f_{\alpha\beta}$ in the projection

$$(9.12) \qquad 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow SL_2(\mathbb{C}) \longrightarrow PSL_2(\mathbb{C}) \longrightarrow 0,$$

which determines the fractional linear transformation. A choice of $\pm$ on each $U_\alpha \cap U_\beta$ is an element of $H^1(C, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$, indicating that there are 2^{2g} choices of a lift. We make this choice once and for all, and consider $f_{\alpha\beta}$ an $SL_2(\mathbb{C})$-valued 1-cocycle. Since

$$dz_\alpha = \frac{1}{(c_{\alpha\beta} z_\beta + d_{\alpha\beta})^2} \, dz_\beta,$$

a transition function for K_C is given by the cocycle $\left\{(c_{\alpha\beta} z_\beta + d_{\alpha\beta})^2\right\}$ on each $U_\alpha \cap U_\beta$. A *theta characteristic* (or a *spin structure*) $K_C^{\frac{1}{2}}$ is the line bundle defined by the 1-cocycle

$$(9.13) \qquad \xi_{\alpha\beta} = c_{\alpha\beta} z_\beta + d_{\alpha\beta}.$$

Here again, we have 2^{2g} choices $\pm \xi_{\alpha\beta}$ for a transition function of $K_C^{\frac{1}{2}}$. Since we have already made a choice of the sign for $f_{\alpha\beta}$, we have a consistent choice in (9.13), as explained below. Thus we see that the choice of the lift (9.12) is determined by $K_C^{\frac{1}{2}}$. From (9.13), we obtain

$$(9.14) \qquad \partial_\beta^2 \xi_{\alpha\beta} = 0.$$

This property plays an essential role in our construction of global connections on C. First we show that actually (9.14) implies (9.11).

Proposition 9.2 (A condition for projective coordinate)**.** *A coordinate system of C with which the second derivative of the transition function of $K_C^{\frac{1}{2}}$ vanishes is a projective coordinate system.*

Proof. The condition (9.14) means that the transition function of $K_C^{\frac{1}{2}}$ is a linear polynomial $c_{\alpha\beta}z_\beta + d_{\alpha\beta}$ satisfying the cocycle condition. Therefore,

$$\frac{dz_\alpha}{dz_\beta} = \frac{1}{(c_{\alpha\beta}z_\beta + d_{\alpha\beta})^2}.$$

Solving this differential equation, we obtain

$$(9.15) \qquad z_a = \frac{m_{\alpha\beta}(c_{\alpha\beta}z_\beta + d_{\alpha\beta}) - 1/c_{\alpha\beta}}{c_{\alpha\beta}z_\beta + d_{\alpha\beta}}$$

with a constant of integration $m_{\alpha\beta}$. In this way we find an element of $SL_2(\mathbb{C})$ on each $U_\alpha \cap U_\beta$. The cocycle condition makes (9.15) exactly (9.11). $\qquad\square$

The transition function $f_{\alpha\beta}$ defines a rank 2 vector bundle E on C whose structure group is $SL_2(\mathbb{C})$. Since $f_{\alpha\beta}$ is a *constant* element of $SL_2(\mathbb{C})$, the notion of locally constant sections of E makes sense independent of the coordinate chart. Thus defining $\nabla_\alpha = d$ on U_α for each U_α determines a global connection ∇ in E. Suppose we have a projective coordinate system on C. Let E be the vector bundle we have just constructed, and $\mathbb{P}(E)$ its projectivization, i.e., the $\mathbb{P}^1$ bundle associated with E. Noticing that $\mathrm{Aut}(\mathbb{P}^1) = PSL_2(\mathbb{C})$, the local coordinate system $\{z_\alpha\}$ of (9.11) is a global section of $\mathbb{P}(E)$. Indeed, this section defines a map

$$(9.16) \qquad z_\alpha : U_\alpha \longrightarrow \mathbb{P}^1,$$

which induces the projective structure of $\mathbb{P}^1$ into U_α via pull back. The map (9.16) is not a constant, because its derivative dz_α never vanishes on the intersection $U_\alpha \cap U_\beta$. A global section of $\mathbb{P}(E)$ corresponds to a line subbundle F of E, such that E is realized as an extension $0 \longrightarrow F \longrightarrow E \longrightarrow E/F \longrightarrow 0$. The equality

$$\begin{bmatrix} z_a \\ 1 \end{bmatrix} = \frac{1}{c_{\alpha\beta}z_\beta + d_{\alpha\beta}} \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \begin{bmatrix} z_\beta \\ 1 \end{bmatrix}$$

shows that $\begin{bmatrix} z_a \\ 1 \end{bmatrix}$ defines a global section of the vector bundle $K_C^{-\frac{1}{2}} \otimes E$, and that the projectivization image of this section is the global section $\{z_\alpha\}$ of $\mathbb{P}(E)$ corresponding to F. Here, the choice of the theta characteristic is consistently made so that the $\pm$ ambiguity of (9.13) and the one in the lift of the fractional linear transformation to $f_{\alpha\beta} \in SL_2(\mathbb{C})$ cancel. Since this section is nowhere vanishing, it generates a trivial subbundle

$$\mathcal{O}_C = K_C^{-\frac{1}{2}} \otimes F \subset K_C^{-\frac{1}{2}} \otimes E.$$

Therefore, $F = K_C^{\frac{1}{2}}$. Note that $\det E = \mathcal{O}_C$, hence $E/F = K_C^{-\frac{1}{2}}$, and E is an extension

$$(9.17) \qquad 0 \longrightarrow K_C^{\frac{1}{2}} \longrightarrow E \longrightarrow K_C^{-\frac{1}{2}} \longrightarrow 0,$$

determining an element of $\mathrm{Ext}^1\big(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}}\big) \cong H^1(C, K_C) \cong \mathbb{C}$.

There is a more straightforward way to obtain (9.17).

Theorem 9.3 (Projective coordinate systems and opers)**.** *Every projective coordinate system* (9.11) *determines an oper* (E, ∇) *of Definition 9.1 with* $F = K_C^{\frac{1}{2}}$.

Proof. Let $\sigma_{\alpha\beta} = -(d/dz_\beta)\xi_{\alpha\beta} = -c_{\alpha\beta}$, where $\xi_{\alpha\beta}$ is defined by (9.13). Since

(9.18)
$$g_{\alpha\beta} := \begin{bmatrix} \xi_{\alpha\beta} & \sigma_{\alpha\beta} \\ & \xi_{\alpha\beta}^{-1} \end{bmatrix} = \begin{bmatrix} c_{\alpha\beta}z_\beta + d_{\alpha\beta} & -c_{\alpha\beta} \\ & (c_{\alpha\beta}z_\beta + d_{\alpha\beta})^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} & 1 \\ -1 & z_\alpha \end{bmatrix} \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \begin{bmatrix} z_\beta & -1 \\ 1 & \end{bmatrix},$$

which follows from

$$a_{\alpha\beta} - c_{\alpha\beta}z_\alpha = \frac{a_{\alpha\beta}(c_{\alpha\beta}z_\beta + d_{\alpha\beta}) - c_{\alpha\beta}(a_{\alpha\beta}z_\beta + b_{\alpha\beta})}{(c_{\alpha\beta}z_\beta + d_{\alpha\beta})} = \frac{1}{(c_{\alpha\beta}z_\beta + d_{\alpha\beta})},$$

we find that $f_{\alpha\beta}$ and $g_{\alpha\beta}$ define the same $SL_2(\mathbb{C})$-bundle E. The shape of the matrix $g_{\alpha\beta}$ immediately shows (9.17). Since the connection ∇ in E is simply d on each U_α with respect to $f_{\alpha\beta}$, the differential operator on U_α with respect to the transition function $g_{\alpha\beta}$ is given by

(9.19)
$$\nabla_\alpha := \begin{bmatrix} & 1 \\ -1 & z_\alpha \end{bmatrix} d \begin{bmatrix} z_\alpha & -1 \\ 1 & \end{bmatrix} = d - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} dz_\alpha.$$

Since the $(2,1)$-component of the connection matrix is dz_α which is nowhere vanishing,

(9.20)
$$F = K_C^{\frac{1}{2}} \xrightarrow{\nabla} E \otimes K_C \longrightarrow (E/F) \otimes K_C \cong K_C^{\frac{1}{2}}$$

given by this component is an isomorphism. This proves that (E, ∇) is an $SL_2(\mathbb{C})$-oper. $\square$

In the final step of the proof to show (E, ∇) is an oper, we need that (9.20) is an $\mathcal{O}_C$-linear homomorphism. This is because we are considering the difference of connections ∇ and $\nabla|_F$ in E. More generally, suppose we have two connections ∇_1 and ∇_2 in the same vector bundle E. Then the Leibniz condition tells us that

$$\nabla_1(fs) - \nabla_2(fs) = f\nabla_1(s) - f\nabla_2(s)$$

for $f \in \mathcal{O}_C$ and $s \in E$. Therefore, $\nabla_1 - \nabla_2 : E \longrightarrow E \otimes K_C$ is an $\mathcal{O}_C$-module homomorphism. Although the extension class of (9.17) is parameterized by $H^1(C, K_C) = \mathbb{C}$, the complex structure of E depends only if $\sigma_{\alpha\beta} = 0$ or not. This is because

(9.21)
$$\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix} \begin{bmatrix} \xi_{\alpha\beta} & \sigma_{\alpha\beta} \\ & \xi_{\alpha\beta}^{-1} \end{bmatrix} \begin{bmatrix} \lambda^{-1} & \\ & \lambda \end{bmatrix} = \begin{bmatrix} \xi_{\alpha\beta} & \lambda^2\sigma_{\alpha\beta} \\ & \xi_{\alpha\beta}^{-1} \end{bmatrix},$$

hence we can normalize $\sigma_{\alpha\beta} = 0$ or $\sigma_{\alpha\beta} = 1$. The former case gives the trivial extension of two line bundles $E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$. Since $\sigma_{\alpha\beta} = c_{\alpha\beta} = 0$, the projective coordinate system (9.11) is actually an *affine* coordinate system. Since we are assuming $g(C) > 1$, there is no affine structure in C. Therefore, only the latter case can happen. And the latter case gives a non-trivial extension, as we will show later.

Suppose we have another projective structure in C subordinating the same complex structure of C. Then we can adjust the $\pm$ signs of the lift of (9.12) and the square root of (9.13) so that we obtain the exact same holomorphic vector bundle E of (9.17). Since we are dealing with a different coordinate system, the only change we have is reflected in the connection ∇. Thus two different projective structures give rise to two connections in the same vector bundle E. Hence this difference is an $\mathcal{O}_C$-linear homomorphism $E \longrightarrow E \otimes K_C$ as noted above. This consideration motivates the following.

A **Higgs bundle** of rank r [55, 56] defined on C is a pair (E, ϕ) consisting of a holomorphic vector bundle E of rank r on C and an $\mathcal{O}_C$-module homomorphism

$$\phi : E \longrightarrow E \otimes K_C.$$

An $SL_2(\mathbb{C})$-Higgs bundle is a pair (E, ϕ) of rank 2 with a fixed isomorphism $\det E = \mathcal{O}_C$ and $\operatorname{tr}\phi = 0$. It is *stable* if every line subbundle $F \subset E$ that is invariant with respect to ϕ, i.e., $\phi : F \longrightarrow F \otimes K_C$, has a negative degree $\deg F < 0$. The moduli spaces of stable Higgs bundles are constructed in [100]. We denote by $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of stable holomorphic $SL_2(\mathbb{C})$-Higgs bundles on C. It is diffeomorphic to the moduli space $\mathcal{M}_{\mathrm{deR}}$ of pairs (E, ∇) consisting of an irreducible holomorphic connection in an $SL_2(\mathbb{C})$-bundle (see [24, 55, 100]). A particular diffeomorphism

$$(9.22) \qquad \nu : \mathcal{M}_{\mathrm{Dol}} \xrightarrow{\sim} \mathcal{M}_{\mathrm{deR}}$$

is the **non-Abelian Hodge correspondence**, which is explained in Section 11.

The total space of the line bundle K_C is the cotangent bundle $\pi : T^*C \longrightarrow C$ of the curve C. We denote by $\eta \in H^0(T^*C, \pi^*K_C)$ the tautological section

$$
\begin{array}{ccc}
 & \pi^*K_C & K_C \\
{\scriptstyle \eta}\nearrow & \downarrow & \downarrow \\
T^*C \xrightarrow{\;=\;} T^*C & \xrightarrow{\;\pi\;} & C,
\end{array}
$$

which is a holomorphic 1-form on T^*C. Since ϕ is an $\operatorname{End}(E)$-valued holomorphic 1-form on C, its eigenvalues are 1-forms. The set of eigenvalues is thus a multivalued section of K_C, and hence a multivalued section of $\pi : T^*C \longrightarrow C$. The image $\Sigma \subset T^*C$ of this multivalued section is the **Hitchin spectral curve**, which defines a ramified covering of C. The formal definition of Hitchin spectral curve Σ is that it is the divisor of zeros in T^*C of the characteristic polynomial

$$\det(\eta - \pi^*\phi) \in \pi^*K_C^{\otimes 2}.$$

Hitchin fibration [55] is a holomorphic fibration

$$(9.23) \qquad \mu_H : \mathcal{M}_{\mathrm{Dol}} \ni (E, \phi) \longmapsto \det(\eta - \pi^*\phi) \in B, \qquad B := H^0\left(C, K_C^{\otimes 2}\right),$$

that defines an algebraically completely integrable Hamiltonian system in $\mathcal{M}_{\mathrm{Dol}}$. Hitchin notices in [55] that the choice of a spin structure $K_C^{\frac{1}{2}}$ that we have made allows us to construct a natural section $\kappa : B \hookrightarrow \mathcal{M}_{\mathrm{Dol}}$. Define

$$(9.24) \qquad X_- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad X_+ := X_-^t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad H := [X_+, X_-] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These elements generate the Lie algebra $\langle X_+, X_-, H \rangle \cong sl_2(\mathbb{C})$.

Lemma 9.4. *Let* $q \in B = H^0(C, K_C^{\otimes 2})$ *be an arbitrary point of the Hitchin base* B, *and define a Higgs bundle* $(E_0, \phi(q))$ *consisting of a vector bundle*

$$(9.25) \qquad E_0 := \left(K_C^{\frac{1}{2}}\right)^{\otimes H} = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$$

and a Higgs field

$$(9.26) \qquad \phi(q) := X_- + qX_+ = \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix}.$$

Then it is a stable $SL_2(\mathbb{C})$-*Higgs bundle. The Hitchin section is defined by*

$$(9.27) \qquad \kappa : B \ni q \longmapsto (E_0, \phi(q)) \in \mathcal{M}_{\mathrm{Dol}},$$

which gives a biholomorphic map between B *and* $\kappa(B) \subset \mathcal{M}_{\mathrm{Dol}}$.

Proof. We first note that $X_- : E_0 \longrightarrow E_0 \otimes K_C$ is a globally defined $\mathrm{End}_0(E_0)$-valued 1-form, since it is essentially the constant map

$$(9.28) \qquad 1 : K_C^{\frac{1}{2}} \xrightarrow{\;=\;} K_C^{-\frac{1}{2}} \otimes K_C.$$

Similarly, multiplication by a quadratic differential gives

$$q : K_C^{-\frac{1}{2}} \longrightarrow K_C^{\frac{3}{2}} = K_C^{\frac{1}{2}} \otimes K_C.$$

Thus $\phi(q) : E_0 \longrightarrow E_0 \otimes K_C$ is globally defined as a Higgs field in E_0. The Higgs pair is stable because no line subbundle of E_0 is invariant under $\phi(q)$, unless $q = 0$. And when $q = 0$, the invariant line subbundle $K_C^{\frac{1}{2}}$ has degree $g - 1$, which is positive since $g \geq 2$. $\qquad \square$

Remark 9.5. Hitchin sections exist for the moduli space of stable G-Higgs bundles for an arbitrary simple complex algebraic group G. The construction utilizes Kostant's *principal three-dimensional subgroup* (TDS) [70]. The use of TDS is crucial in our quantization, as noted in [27, 32].

The image $\kappa(B)$ is a holomorphic Lagrangian submanifold of a holomorphic symplectic space $\mathcal{M}_{\mathrm{Dol}}$. The holomorphic symplectic structure of $\mathcal{M}_{\mathrm{Dol}}$ is induced from its open dense subspace $T^*\mathcal{SU}(2, C)$, where $\mathcal{SU}(2, C)$ is the moduli space of rank 2 stable bundles of degree 0 on C. Since the codimension of the complement of $T^*\mathcal{SU}(2, C)$ in $\mathcal{M}_{\mathrm{Dol}}$ is 2, the natural holomorphic symplectic form on the cotangent bundle automatically extends to $\mathcal{M}_{\mathrm{Dol}}$.

Our first step of constructing the quantization of the Hitchin spectral curve Σ is to define $\hbar$-connections on C that are holomorphically depending on $\hbar$. We use a one-parameter family $\mathcal{E}$ of deformations of vector bundles

$$
\begin{array}{ccc}
E_\hbar & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
C \times \{\hbar\} & \longrightarrow & C \times H^1(C, K_C),
\end{array}
$$

and a $\mathbb{C}$-linear first-order differential operator $\hbar\nabla^\hbar : E_\hbar \longrightarrow E_\hbar \otimes K_C$ depending holomorphically on $\hbar \in H^1(C, K_C) \cong \mathbb{C}$ for $\hbar \neq 0$. Here, we identify the Planck constant $\hbar$ of the quantization as a geometric parameter

$$(9.29) \qquad \hbar \in H^1(C, K_C) = \mathrm{Ext}^1\left(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}} \right) \cong \mathbb{C},$$

which determines a unique extension

$$(9.30) \qquad 0 \longrightarrow K_C^{\frac{1}{2}} \longrightarrow E_\hbar \longrightarrow K_C^{-\frac{1}{2}} \longrightarrow 0.$$

This is exactly the same as (9.17). The extension $E_\hbar$ is given by a system of transition functions

$$(9.31) \qquad E_\hbar \longleftrightarrow \left\{ \begin{bmatrix} \xi_{\alpha\beta} & \hbar\sigma_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{bmatrix} \right\}$$

on each $U_\alpha \cap U_\beta$. The cocycle condition for the transition functions translates into a condition

$$(9.32) \qquad \sigma_{\alpha\gamma} = \xi_{\alpha\beta}\sigma_{\beta\gamma} + \sigma_{\alpha\beta}\xi_{\beta\gamma}^{-1}.$$

The application of the exterior differentiation d to the cocycle condition $\xi_{\alpha\gamma} = \xi_{\alpha\beta}\xi_{\beta\gamma}$ for $K_C^{\frac{1}{2}}$ yields

$$\frac{d\xi_{\alpha\gamma}}{dz_\gamma}dz_\gamma = \frac{d\xi_{\alpha\beta}}{dz_\beta}dz_\beta\xi_{\beta\gamma} + \xi_{\alpha\beta}\frac{d\xi_{\beta\gamma}}{dz_\gamma}dz_\gamma.$$

Noticing $\xi_{\alpha\beta}^2 = \frac{dz_\beta}{dz_\alpha}$, we find that

$$(9.33) \qquad\qquad \sigma_{\alpha\beta} := -\frac{d\xi_{\alpha\beta}}{dz_\beta} = -\partial_\beta\xi_{\alpha\beta}$$

solves (9.32). The negative sign is chosen to relate (9.31) and (9.11). By the same reason as before, the complex structure of the vector bundle $E_\hbar$ is isomorphic to E_1 if $\hbar \neq 0$, and to E_0 of (9.25) if $\hbar = 0$. The transition function can also be written as

$$(9.34) \qquad \begin{bmatrix} \xi_{\alpha\beta} & \hbar\sigma_{\alpha\beta} \\ & \xi_{\alpha\beta}^{-1} \end{bmatrix} = \exp\left(\log\xi_{\alpha\beta}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\exp\left(-\hbar\partial_\beta\log\xi_{\alpha\beta}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

Therefore, in the multiplicative sense, the extension class is determined by $\partial_\beta\log\xi_{\alpha\beta}$.

Lemma 9.6. *The extension class $\sigma_{\alpha\beta}$ of (9.33) defines a non-trivial extension (9.30).*

Proof. The long exact sequences of cohomologies

$$
\begin{array}{ccccccc}
H^1(C,\mathbb{C}) & \longrightarrow & H^1(C,\mathcal{O}_C) & \longrightarrow & H^1(C,K_C) & \xrightarrow{\sim} & H^2(C,\mathbb{C}) \\
\downarrow & & \downarrow & & \| & & \\
H^1(C,\mathbb{C}^*) & \longrightarrow & H^1(C,\mathcal{O}_C^*) & \xrightarrow{d\log} & H^1(C,K_C) & & \\
{\scriptstyle 0}\downarrow & & {\scriptstyle c_1}\downarrow & & & & \\
H^2(C,\mathbb{Z}) & \xrightarrow{=} & H^2(C,\mathbb{Z}) & & & &
\end{array}
$$

associated with exact sequences of sheaves

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O}_C & \xrightarrow{d} & K_C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathcal{O}_C^* & \xrightarrow{d\log} & K_C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

show that the class $\{\sigma_{\alpha\beta}\}$ corresponds to the image of $\{\xi_{ab}\}$ via the map

$$H^1(C,\mathcal{O}_C^*) \xrightarrow{d\log} H^1(C,K_C).$$

If $d\log\{\xi_{\alpha\beta}\} = 0 \in H^1(C,K_C)$, then it comes from a class in $H^1(C,\mathbb{C}^*)$, which is the moduli space of line bundles with holomorphic connections, as explained in [8]. It leads to

a contradiction

$$0 = c_1\big(K_C^{\frac{1}{2}}\big) = g - 1 > 0,$$

because $g(C) \geq 2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 9.7. Gukov and Sułkowski [52] defines an intriguing quantizability condition for a spectral curve in terms of the algebraic K-group $K_2\big(\mathbb{C}(\Sigma)\big)$ of the function field of spectral curve Σ. They relate the quantizability and Bloch regulators of [8].

The class $\{\sigma_{\alpha\beta}\}$ of (9.33) gives a natural isomorphism $H^1(C, K_C) \cong \mathbb{C}$. We identify the deformation parameter $\hbar \in \mathbb{C}$ with the cohomology class $\{\hbar\sigma_{\alpha\beta}\} \in H^1(C, K_C) = \mathbb{C}$. We trivialize the line bundle $K_C^{\otimes 2}$ with respect to a coordinate chart $C = \bigcup_\alpha U_\alpha$, and write $q \in H^0(C, K_C^{\otimes}2)$ as $\{(q)_\alpha\}$ that satisfies the transition relation

$$(9.35) \qquad\qquad\qquad\qquad (q)_\alpha = (q)_\beta \xi_{\alpha\beta}^4.$$

The transition function of the trivial extension E_0 is given by

$$(9.36) \qquad\qquad\qquad\qquad \xi_{\alpha\beta}^H = \exp(H \log \xi_{\alpha\beta}).$$

Since $X_- : E_0 \longrightarrow E_0 \otimes K_C$ is a globally defined Higgs filed, its local expressions $\{X_- dz_\alpha\}$ with respect to a coordinate system satisfies the transition relation

$$(9.37) \qquad\qquad X_- dz_\alpha = \exp(H \log \xi_{\alpha\beta}) X_- dz_\beta \exp(-H \log \xi_{\alpha\beta})$$

on every $U_\alpha \cap U_\beta$. The same relation holds for the Higgs field $\phi(q)$ as well:

$$(9.38) \qquad\qquad \phi_\alpha(q) dz_\alpha = \exp(H \log \xi_{\alpha\beta}) \phi_\beta(q) dz_\beta \exp(-H \log \xi_{\alpha\beta}).$$

Theorem 9.8 (Construction of $SL_2(\mathbb{C})$-opers). *On each $U_\alpha \cap U_\beta$ define a transition function*

$$(9.39) \quad g_{\alpha\beta}^\hbar := \exp(H \log \xi_{\alpha\beta}) \exp\big(-\hbar\partial_\beta \log \xi_{\alpha\beta} X_+\big) = \begin{bmatrix} \xi_{\alpha\beta} & \\ & \xi_{\alpha\beta}^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\hbar\partial_\beta \log \xi_{\alpha\beta} \\ & 1 \end{bmatrix},$$

where $\partial_\beta = \frac{d}{dz_\beta}$, and $\hbar\partial_\beta \log \xi_{\alpha\beta} \in H^1(C, K_C)$. Then

- *The collection $\{g_{\alpha\beta}^\hbar\}$ satisfies the cocycle condition*

$$(9.40) \qquad\qquad\qquad\qquad g_{\alpha\beta}^\hbar g_{\beta\gamma}^\hbar = g_{\alpha\gamma}^\hbar,$$

  which defines the holomorphic vector bundle bundle $E_\hbar$ of (9.30).
- *The local expression*

$$(9.41) \qquad\qquad\qquad\qquad \nabla_\alpha^\hbar(0) := d - \frac{1}{\hbar} X_- dz_a$$

  on U_α for $\hbar \neq 0$ defines a global holomorphic connection in $E_\hbar$, i.e.,

$$(9.42) \qquad\qquad d - \frac{1}{\hbar} X_- dz_\alpha = g_{\alpha\beta}^\hbar \left(d - \frac{1}{\hbar} X_- dz_\beta \right) \left(g_{\alpha\beta}^\hbar \right)^{-1},$$

  if and only if the coordinate is a projective coordinate system. We choose one.
- *With this particular projective coordinate system, every point $(E_0, \phi(q)) \in \kappa(B) \subset \mathcal{M}_{\mathrm{Dol}}$ of the Hitchin section (9.27) gives rise to a one-parameter family of $SL_2(\mathbb{C})$-opers $\big(E_\hbar, \nabla^\hbar(q)\big) \in \mathcal{M}_{\mathrm{deR}}$. In other words, the local expression*

$$(9.43) \qquad\qquad\qquad\qquad \nabla_\alpha^\hbar(q) := d - \frac{1}{\hbar} \phi_\alpha(q) dz_\alpha$$

  on every U_α for $\hbar \neq 0$ determines a global holomorphic connection

$$(9.44) \qquad\qquad\qquad\qquad \nabla_\alpha^\hbar(q) = g_{\alpha\beta}^\hbar \nabla_\beta^\hbar(q) \left(g_{\alpha\beta}^\hbar \right)^{-1}$$

       in $E_\hbar$ satisfying the oper condition.
- *Deligne's $\hbar$-connection*

$$\hbar \nabla^\hbar(q) : E_\hbar \longrightarrow E_\hbar \otimes K_C \tag{9.45}$$

       interpolates the Higgs pair and the oper, i.e., at $\hbar = 0$, the family (9.45) gives the Higgs pair $(E, -\phi(q)) \in \mathcal{M}_{\mathrm{Dol}}$, and at $\hbar = 1$ it gives an $SL_2(\mathbb{C})$-oper $\left(E_1, \nabla^1(q)\right) \in \mathcal{M}_{\mathrm{deR}}$.
- *After a suitable gauge transformation depending on $\hbar$, the $\hbar \to \infty$ limit of the oper $\nabla^\hbar(q)$ exists and is equal to $\nabla^{\hbar=1}(0)$. This point corresponds to the $\mathbb{C}^*$-fixed point on the Hitchin section.*

Proof. The cocycle condition of $g_{\alpha\beta}$ has been established in (9.33) and (9.34). Proof of (9.42) is a straightforward calculation, using the power series expansion of the adjoint action

$$e^{\hbar A} B e^{-\hbar A} = \sum_{n=0}^{\infty} \frac{1}{n!} \hbar^n (\mathrm{ad}_A)^n(B) := \sum_{n=0}^{\infty} \frac{1}{n!} \hbar^n \overbrace{[A, [A, [\cdots, [A, B] \cdots]]]}^{n}. \tag{9.46}$$

It follows that

$$
\begin{aligned}
&g_{\alpha\beta}^\hbar X_- \left(g_{\alpha\beta}^\hbar\right)^{-1} \\
&= \exp(H \log \xi_{\alpha\beta}) \exp\left(-\hbar \partial_\beta \log \xi_{\alpha\beta} X_+\right) X_- \exp\left(\hbar \partial_\beta \log \xi_{\alpha\beta} X_+\right) \exp(-H \log \xi_{\alpha\beta}) \\
&= \exp(H \log \xi_{\alpha\beta}) X_- \exp(-H \log \xi_{\alpha\beta}) - \hbar \partial_\beta \log \xi_{\alpha\beta} H \\
&\quad - \hbar^2 (\partial_\beta \log \xi_{\alpha\beta})^2 \exp(H \log \xi_{\alpha\beta}) X_+ \exp(-H \log \xi_{\alpha\beta}).
\end{aligned}
$$

Note that (9.14) is equivalent to

$$\partial_\beta \partial_\beta \log \xi_{\alpha\beta} = \partial_\beta \left(\xi_{\alpha\beta}^{-1} \partial_\beta \xi_{\alpha\beta}\right) = -\xi_{\alpha\beta}^{-2} (\partial_\beta \xi_{\alpha\beta})^2 = -(\partial_\beta \log \xi_{\alpha\beta})^2,$$

hence to

$$\partial_\beta g_{\alpha\beta}^\hbar \left(g_{\alpha\beta}^\hbar\right)^{-1} = \partial_\beta \log \xi_{\alpha\beta} H + \hbar (\partial_\beta \log \xi_{\alpha\beta})^2 \exp(H \log \xi_{\alpha\beta}) X_+ \exp(-H \log \xi_{\alpha\beta}).$$

Therefore, noticing (9.37), (9.14) is equivalent to

$$\left(\frac{1}{\hbar} g_{\alpha\beta}^\hbar X_- \left(g_{\alpha\beta}^\hbar\right)^{-1} + \partial_\beta g_{\alpha\beta}^\hbar \left(g_{\alpha\beta}^\hbar\right)^{-1}\right) dz_\beta = \frac{1}{\hbar} \exp(H \log \xi_{\alpha\beta}) X_- dz_\beta \exp(-H \log \xi_{\alpha\beta})$$

$$= \frac{1}{\hbar} X_- dz_\alpha.$$

The statement follows from Proposition 9.2.

To prove (9.44), we need, in addition to (9.42), the following relation:

$$(q)_\alpha X_+ dz_\alpha = g_{\alpha\beta}^\hbar (q)_\beta X_+ dz_\beta \left(g_{\alpha\beta}^\hbar\right)^{-1}. \tag{9.47}$$

But (9.47) is obvious from (9.38) and (9.39).

The line bundle F required in the definition of $SL_2(\mathbb{C})$-oper is simply $K_C^{\frac{1}{2}}$. The isomorphism (9.10) is a consequence of (9.28). Finally, the gauge transformation of $\nabla^\hbar(q)$ by a bundle automorphism

$$\hbar^{-\frac{H}{2}} = \begin{bmatrix} \hbar^{-\frac{1}{2}} & \\ & \hbar^{\frac{1}{2}} \end{bmatrix} \tag{9.48}$$

on each coordinate neighborhood U_α gives

$$(9.49) \qquad d - \frac{1}{\hbar}\phi(q) \longmapsto \hbar^{-\frac{H}{2}}\left(d - \frac{1}{\hbar}\phi(q)\right)\hbar^{\frac{H}{2}} = d - \left(X_- + \frac{q}{\hbar^2}X_+\right).$$

This is because

$$\hbar^{-\frac{H}{2}}X_-\hbar^{\frac{H}{2}} = \hbar X_- \qquad \text{and} \qquad \hbar^{-\frac{H}{2}}X_+^2\hbar^{\frac{H}{2}} = \hbar^{-2}X_+^2,$$

which follows from the adjoint formula (9.46). Therefore,

$$\lim_{\hbar\to\infty} \nabla^\hbar(q) \sim d - X_- = \nabla^{\hbar=1}(0),$$

where the symbol $\sim$ means gauge equivalence. This completes the proof of the theorem. $\square$

The construction theorem yields the following.

Theorem 9.9 (Biholomorphic quantization of Hitchin spectral curves). *Let C be a compact Riemann surface of genus $g \geq 2$ with a chosen projective coordinate system subordinating its complex structure. We denote by $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of stable holomorphic $SL_2(\mathbb{C})$-Higgs bundles over C, and by $\mathcal{M}_{\mathrm{deR}}$ the moduli space of irreducible holomorphic $SL_2(\mathbb{C})$-connections on C. For a fixed theta characteristic $K_C^{\frac{1}{2}}$, we have a Hitchin section $\kappa(B) \subset \mathcal{M}_{\mathrm{Dol}}$ of (9.27). We denote by $Op \subset \mathcal{M}_{\mathrm{deR}}$ the moduli space of $SL_2(\mathbb{C})$-opers with the condition that the required line bundle is given by $F = K_C^{\frac{1}{2}}$. Then the map*

$$(9.50) \qquad \mathcal{M}_{\mathrm{Dol}} \supset \kappa(B) \ni (E_0, \phi(q)) \overset{\gamma}{\longmapsto} \left(E_\hbar, \nabla^\hbar(q)\right) \in Op \subset \mathcal{M}_{\mathrm{deR}}$$

evaluated at $\hbar = 1$ is a biholomorphic map with respect to the natural complex structures induced from the ambient spaces.

The biholomorphic quantization (9.50) is also $\mathbb{C}^*$-equivariant. The $\lambda \in \mathbb{C}^*$ action on the Hitchin section is defined by $\phi \longmapsto \lambda\phi$. The oper corresponding to $(E_0, \lambda\phi(q)) \in \kappa(B)$ is $d - \frac{\lambda}{\hbar}\phi(q)$.

Proof. The $\mathbb{C}^*$-equivariance follows from the same argument of the gauge transformation (9.48), (9.49). The action $\phi \longmapsto \lambda\phi$ on the Hitchin section induces a weighted action

$$B \ni q \longmapsto \lambda^2 q \in B$$

through μ_H. Then we have the gauge equivalence via the gauge transformation $\left(\frac{\lambda}{\hbar}\right)^{\frac{H}{2}}$:

$$d - \frac{\lambda}{\hbar}\phi(q) \sim \left(\frac{\lambda}{\hbar}\right)^{\frac{H}{2}}\left(d - \frac{\lambda}{\hbar}\phi(q)\right)\left(\frac{\lambda}{\hbar}\right)^{-\frac{H}{2}} = d - \left(X_- + \frac{\lambda^2 q}{\hbar^2}X_+\right).$$

$\square$

Remark 9.10. In the construction theorem, our use of a projective coordinate system is essential, through (9.13). Only in such a coordinate, our particular definition (9.43) makes sense. This is due to the vanishing of the second derivative of $\xi_{\alpha\beta}$. And as we have seen above, the projective coordinate system determines the *origin* $\nabla^1(0)$ of the space Op of opers. Other opers are simply translation $\nabla^1(q)$ from the origin by $q \in H^0(C, K_C^{\otimes 2})$.

10. Semi-classical limit of $SL_2(\mathbb{C})$-opers

A holomorphic connection on a compact Riemann surface C is automatically flat. There-
fore, it defines a $\mathcal{D}$-module over C. Continuing the last section's conventions, let us fix
a projective coordinate system on C, and let $(E_0, \phi(q)) = \kappa(q)$ be a point on the Hitchin
section of (9.27). It uniquely defines an $\hbar$-family of opers $\big(E_\hbar, \nabla^\hbar(q)\big)$.

In this section, we establish that the $\hbar$-connection $\hbar\nabla^\hbar(q)$ defines a family of $\mathcal{D}$-modules
on C parametrized by B such that the semi-classical limit of the family agrees with the
family of spectral curves over B.

To calculate the semi-classical limit, let us trivialize the vector bundle $E_\hbar$ on each simply
connected coordinate neighborhood U_α with coordinate z_α of the chosen projective coordi-
nate system. A flat section Ψ_α of $E_\hbar$ over U_α is a solution of

$$(10.1) \qquad \hbar\nabla^\hbar_\alpha(q)\Psi_\alpha := (\hbar d - \phi_\alpha(q)) \begin{bmatrix} \hbar\psi \\ \psi \end{bmatrix}_\alpha = 0,$$

with an appropriate unknown function ψ. Since $\Psi_\alpha = g^\hbar_{\alpha\beta}\Psi_\beta$, the function ψ on U_α satisfies
the transition relation $(\psi)_\alpha = \xi^{-1}_{\alpha\beta}(\psi)_\beta$. It means that ψ is actually a local section of the
line bundle $K_C^{-\frac{1}{2}}$. There are two linearly independent solutions of (10.1), because q is a
holomorphic function on U_α. Since $\phi(q)$ is independent of $\hbar$ and takes the form

$$(10.2) \qquad \phi(q) = \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix},$$

we see that (10.1) is equivalent to the second order equation

$$(10.3) \qquad \hbar^2\psi'' - q\psi = 0$$

for $\psi \in K_C^{-\frac{1}{2}}$. Since we are using a fixed projective coordinate system, the connection $\nabla^\hbar(q)$
takes the same form on each coordinate neighborhood U_α. Therefore, the shape of the
differential equation of (10.3) as an equation for ψ is again the same on every coordinate
neighborhood, as we wished to achieve in (9.2). This is exactly what we refer to as the
quantum curve of the spectral curve $\det(\eta - \phi(q)) = 0$. It is now obvious to calculate the
semi-classical limit of the $\mathcal{D}$-module corresponding to $\hbar\nabla^\hbar(q)$.

Theorem 10.1 (Semi-classical limit of an oper). *Under the same setting of Theorem 9.9, let
$\mathcal{E}(q)$ denote the $\mathcal{D}$-module $\big(E_\hbar, \hbar\nabla^\hbar(q)\big)$ associated with the oper of (9.50). Then the semi-
classical limit of $\mathcal{E}(q)$ is the spectral curve $\Sigma \subset T^*C$ of $\phi(q)$ defined by the characteristic
equation $\det(\eta - \phi(q)) = 0$.*

The semi-classical limit of (10.3) is the limit

$$(10.4) \qquad \lim_{\hbar\to 0} e^{-\frac{1}{\hbar}S_0(z_\alpha)} \left[\hbar^2 \left(\frac{d}{dz_\alpha} \right)^2 - q \right] e^{\frac{1}{\hbar}S_0(z_a)} = y^2 - q,$$

where $S_0(z_\alpha)$ is a holomorphic function on U_α so that $dS_0 = ydz_\alpha$ gives a local trivialization
of T^*C over U_α. The computation of semi-classical limit is the same as the calculation of
the determinant of the connection $\hbar\nabla^\hbar(q)$, after taking conjugation by the scalar diagonal
matrix $e^{-\frac{1}{\hbar}S_0(z_\alpha)}I_{2\times 2}$, and then take the limit as $\hbar$ goes to 0.

For every $\hbar \in H^1(C, K_C)$, the $\hbar$-connection $\big(E_\hbar, \hbar\nabla^\hbar(q)\big)$ of (9.45) defines a global $\mathcal{D}_C$-
module structure in $E_\hbar$. Thus we have constructed a universal family $\mathcal{E}_C$ of $\mathcal{D}_C$-modules on

a given C with a fixed spin structure and a projective structure:

(10.5)
$$
\begin{array}{ccc}
\mathcal{E}_C & \xleftarrow{\quad\supset\quad} & \left(E_\hbar, \nabla^\hbar(q)\right) \\
\downarrow & & \downarrow \\
C \times B \times H^1(C, K_C) & \xleftarrow{\quad\quad} & C \times \{q\} \times \{\hbar\}
\end{array}
$$

The universal family $\mathcal{S}_C$ of spectral curves is defined over $C \times B$.

(10.6)
$$
\begin{array}{ccccc}
\mathbb{P}\left(K_C \oplus \mathcal{O}_C\right) \times B & \xleftarrow{\quad\quad} & \mathcal{S}_C & \xleftarrow{\quad\quad} & \left(\det\left(\eta - \phi(q)\right)\right)_0 \\
\downarrow & & \downarrow & & \downarrow \\
C \times B & \xleftarrow{\quad=\quad} & C \times B & \xleftarrow{\quad\supset\quad} & C \times \{q\}.
\end{array}
$$

The semi-classical limit is thus a map of families

(10.7)
$$
\begin{array}{ccc}
\mathcal{E}_C & \xrightarrow{\quad\quad} & \mathcal{S}_C \\
\downarrow & & \downarrow \\
C \times B \times H^1(C, K_C) & \xrightarrow{\quad\quad} & C \times B.
\end{array}
$$

11. Non-Abelian Hodge correspondence between Hitchin moduli spaces

The biholomorphic map (9.50) is defined by fixing a projective structure of the base curve C. Gaiotto [43] conjectured that such a correspondence would be canonically constructed through a *scaling limit* of non-Abelian Hodge correspondence. The conjecture has been solved in [27] for *holomorphic* Hitchin moduli spaces $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{deR}}$ constructed over an arbitrary complex simple and simply connected Lie group G. In this section, we review the main result of [27] for $G = SL_2(\mathbb{C})$ and compare it with our quantization.

We denote by E^{top} the topologically trivial complex vector bundle of rank 2 on a compact Riemann surface C of genus $g \geq 2$. The correspondence between stability conditions of holomorphic vector bundles on C and PDEs on differential geometric data is used in Narasimhan-Seshadri [90] to obtain topological structures of the moduli space of stable bundles (see also [5, 88]). Extending this classical case, the stability condition for an $SL_2(\mathbb{C})$-Higgs bundle (E, ϕ) translates into a system of PDEs, known as *Hitchin's equations*, imposed on a set of geometric data [24, 55, 100]. The data we need are a Hermitian fiber metric h on E^{top}, a unitary connection D in E^{top} with respect to h, and a differentiable $sl_2(\mathbb{C})$-valued 1-form ϕ on C. In this section we use D for unitary connections to avoid confusion with holomorphic connections we have been using until the last section. **Hitchin's equations** are the following system of nonlinear PDEs.

(11.1)
$$
\begin{cases}
F_D + [\phi, \phi^\dagger] = 0 \\
D^{0.1}\phi = 0.
\end{cases}
$$

Here, F_D denotes the curvature 2-form of D, $\phi^\dagger$ is the Hermitian conjugate of ϕ with respect to the metric h, and $D^{0,1}$ is the Cauchy-Riemann part of D defined by the complex structure of the base curve C. $D^{0,1}$ determines a complex structure in E^{top}, which we simply denote by E. Then ϕ becomes a holomorphic Higgs field in E because it satisfies the Cauchy-Riemann equation (11.1). The pair (E, ϕ) constructed in this way from a solution of Hitchin's equations is a *stable* Higgs bundle. Conversely [100], a stable Higgs bundle (E, ϕ) gives rise to a unique harmonic Hermitian metric h and the Chern connection D

with respect to h so that the data satisfy Hitchin's equations. The stability condition for the holomorphic Higgs pair (E, ϕ) is thus translated into (11.1).

Define a one-parameter family of connections

$$(11.2) \qquad D(\zeta) := \frac{1}{\zeta} \cdot \phi + D + \zeta \cdot \phi^\dagger, \qquad \zeta \in \mathbb{C}^*.$$

Then the flatness of $D(\zeta)$ for all ζ is equivalent to (11.1). The *non-Abelian Hodge correspondence* [24, 55, 78, 100] is a diffeomorphic correspondence

$$\nu : \mathcal{M}_{\text{Dol}} \ni (E, \phi) \longmapsto (\widetilde{E}, \widetilde{\nabla}) \in \mathcal{M}_{\text{deR}}.$$

Proving the diffeomorphism of these moduli spaces is far beyond the scope of this article. Here, we only give the definition of the map ν. We start with the solution (D, ϕ, h) of Hitchin's equations corresponding to a stable Higgs bundle (E, ϕ). It induces a family of flat connections $D(\zeta)$. Define a complex structure $\widetilde{E}$ in E^{top} by $D(\zeta = 1)^{0,1}$. Since $D(\zeta)$ is flat, $\widetilde{\nabla} := D(\zeta = 1)^{1,0}$ is automatically a holomorphic connection in $\widetilde{E}$. Stability of (E, ϕ) implies that the resulting holomorphic connection is irreducible, hence $(\widetilde{E}, \widetilde{\nabla}) \in \mathcal{M}_{\text{deR}}$. Since this correspondence goes through the real unitary connection D, the change of the complex structure of E to that of $\widetilde{E}$ is not a holomorphic deformation.

Extending the idea of one-parameter family (11.2), Gaiotto conjectures:

Conjecture 11.1 (Gaiotto [43]). *Let* (D, ϕ, h) *be the solution of* (11.1) *corresponding to a sable Higgs bundle* $(E_0, \phi(q))$ *on the* $SL_2(\mathbb{C})$*-Hitchin section* (9.27). *Consider the following two-parameter family of connections*

$$(11.3) \qquad D(\zeta, R) := \frac{1}{\zeta} \cdot R\phi + D + \zeta \cdot R\phi^\dagger, \qquad \zeta \in \mathbb{C}^*, R \in \mathbb{R}_+.$$

Then the scaling limit

$$(11.4) \qquad \lim_{\substack{R \to 0, \zeta \to 0 \\ \zeta/R = \hbar}} D(\zeta, R)$$

exists for every $\hbar \in \mathbb{C}^*$, *and forms an* $\hbar$*-family of* $SL_2(\mathbb{C})$*-opers.*

Remark 11.2.   (1) The existence of the limit is non-trivial, because the Hermitian metric h blows up as $R \to 0$.
  (2) Unlike the case of non-Abelian Hodge correspondence, the Gaiotto limit works only for a point in the Hitchin section.

Theorem 11.3 ([27]). *Gaiotto's conjecture holds for an arbitrary simple and simply connected complex algebraic group* G.

Recall that the representation (3.2) gives a realization of C from its universal covering space $\mathbb{H}$ as

$$C \cong \mathbb{H}/\rho\big(\pi_1(C)\big).$$

The representation ρ lifts to $SL_2(\mathbb{R}) \subset SL_2(\mathbb{C})$, and defines a projective structure in C subordinating its complex structure coming from $\mathbb{H}$. This projective structure is what we call the *Fuchsian* projective structure.

Corollary 11.4 (Gaiotto correspondence and quantization [27]). *Under the same setting of Conjecture 11.1, the limit oper of* (11.4) *is given by*

$$(11.5) \qquad \lim_{\substack{R \to 0, \zeta \to 0 \\ \zeta/R = \hbar}} D(\zeta, R) = d - \frac{1}{\hbar}\phi(q) = \nabla^\hbar(q), \qquad \hbar \neq 0,$$

with respect to the Fuchsian projective coordinate system. The correspondence

$$(E_0, \phi(q)) \stackrel{\gamma}{\longmapsto} \left(E_\hbar, \nabla^\hbar(q) \right)$$

is biholomorphic, unlike the non-Abelian Hodge correspondence.

Proof. The key point is that since E_0 is made out of K_C, the fiber metric h naturally comes from the metric of C itself. Hitchin's equations (11.1) for $q = 0$ then become a harmonic equation for the metric of C, and its solution is given by the constant curvature hyperbolic metric within the conformal class of the initial metric. This metric in turn defines the Fuchsian projective structure in C. For more detail, we reefer to [26, 27]. $\square$

Remark 11.5. The case of an arbitrary simple algebraic group G of the conjecture, and the whole story of Part 2 for G, have been worked out in [27, 32]. The key point is to use Kostant's TDS of [70] and transcribe the $SL_2(\mathbb{C})$ situation into G.

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