GRAPHICAL EXPANSION OF NON-COMMUTATIVE MATRIX INTEGRALS

MOTOHICO MULASE

1. Yesterday we learned about many great achievements of Professor Harold Widom from the speakers of the conference. I believe it is fair to say Harold has been someone like King Midas throughout his mathematical career: the only difference is that whatever he touched did not change to gold; they turned into papers, truly monumental papers.

The impact of Random Matrix Theory is felt in every corner of mathematics. It is my pleasure to report to you today about such an impact in finite group theory. Using ideas from RMT, one can obtain a very simple and beautiful formula in finite group theory.

Of course there are extremely deep works on symmetric groups in the context of RMT, due to many of you in the audience. What I'd like to discuss today is not directly related these works. Rather, I would like to talk about a theorem that is applicable to all finite groups.

This talk is based on my recent collaborations with Michael Penkava, Andrew Waldron, and Josephine Yu.

2. Let me begin with reviewing some classical formulas in finite group theory. So let G be a finite group, and $\hat{G}$ the set of all complex irreducible representations of G. Then we have

$$\sum_{\lambda\in\hat{G}}(\dim\lambda)^2 = |G|$$
$$\sum_{\lambda\in\hat{G}}(\dim\lambda)^0 = |Conj|\ ,$$

where $Conj$ denotes the set of conjugacy classes of G. Since we are considering the power sums of the dimensions of irreducible representations, we may ask what we get when the power is one. There is no general formula for that, but if a group is special such as a symmetric group $\mathfrak{S}_n$, then we have

$$\sum_{\lambda\in\hat{G}}(\dim\lambda)^1 = |Inv|,$$

where Inv is the set of involutions of G. Now we have powers 0, 1, and 2 appearing, why don't we consider more general powers?

Before going on, we need a little more analysis of the set $\hat{G}$ to modify the third formula to make it applicable for all finite groups. We classify complex representations into three categories

$$\hat{G} = \hat{G}_1 \cup \hat{G}_2 \cup \hat{G}_4\ .$$

Date: September 24, 2002.

I believe the RMT community would not ask why I used 4 for the third one. They are defined by

$$\hat{G}_1 = \left\{ \lambda \in \hat{G} \;\middle|\; \frac{1}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^2) = 1 \right\} = GOE$$

$$\hat{G}_2 = \left\{ \lambda \in \hat{G} \;\middle|\; \frac{1}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^2) = 0 \right\} = GUE$$

$$\hat{G}_4 = \left\{ \lambda \in \hat{G} \;\middle|\; \frac{1}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^2) = -1 \right\} = GSE \; .$$

We recall that every irreducible representation of G is naturally unitary. If it happens to be real orthogonal, then it belongs to $\hat{G}_1$. If a representation has a G-invariant symplectic structure, then it is in $\hat{G}_4$. Otherwise, it belongs to $\hat{G}_2$. Now the third formula becomes completely general:

$$\sum_{\lambda \in \hat{G}_1} (\dim \lambda)^1 - \sum_{\lambda \in \hat{G}_4} (\dim \lambda)^1 = |Inv|,$$

which is true for every finite group G. From RMT, we can deduce the following:

Theorem 1 (math.QA/0209008)**.** *Let S be a compact surface without boundary.*

(1) *If S is orientable, then*

$$\sum_{\lambda \in \hat{G}} (\dim \lambda)^{\chi(S)} = |G|^{\chi(S)-1} |\mathrm{Hom}(\pi_1(S), G)| \; .$$

(2) *If S is non-orientable, then*

$$\sum_{\lambda \in \hat{G}_1} (\dim \lambda)^{\chi(S)} + \sum_{\lambda \in \hat{G}_4} (-\dim \lambda)^{\chi(S)} = |G|^{\chi(S)-1} |\mathrm{Hom}(\pi_1(S), G)| \; ,$$

where $\chi(S)$ is the Euler characteristic of S.

First remark I should make here is that the first formula for the orientable case is essentially due to Burnside (1911). I thank Professor Andrei Okounkov for this information. (According to him, Burnside attributes the formula to Frobenius.) In our time, Freed-Quinn (CMP 1993) rediscovered the first formula by using Chern-Simons gauge theory with G as the gauge group. Actually, my talk can be thought of as a mathematical explanation of the recently discovered relation between Chern-Simons gauge theory and RMT. It has to be emphasized, however, that our RMT method works equally well for both orientable and non-orientable surfaces.

From Theorem 1, we can recover the classical formulas I mentioned earlier. For example, if we take S to be a 2-dimensional sphere, then $\chi(S) = 2$ and $\pi_1(S) = 1$, and the first classical formula for the order of the group is recovered. If we chose a real projective plane $S = \mathbb{R}P^2$ for our surface, then $\chi(S) = 1$ and $\pi_1(S) = \{\pm 1\}$, and the homomorphism counts involutions of G. Thus the formula for $|Inv|$ is obtained. Finally, if we take a real 2-torus $S = T^2$, then it is an easy exercise to show

$$|\mathrm{Hom}(\mathbb{Z}^2, G)| = |G| \cdot |Conj| \; ,$$

and hence the second classical formula is recovered.

The key of our generalization of the classical formulas is to identify the power of $\dim \lambda$ as the Euler characteristic of a surface, and replace the right hand side with the number of homomorphisms from the fundamental group of the surface into the group G.

It has to be also remarked that Theorem 1 has a rather simple algebraic proof (see our paper cited above). But we are led to the formula through RMT first, and discovered the simple proof only after formulas are established. RMT also provides the reason why irreducible representations of a finite group has something to do with surface geometry. We note that G and S have no relation what so ever!

3. Now let me turn to the subject appeared in the title of this talk. We need three formulas for graphical expansion of matrix integrals, one each for GOE, GUE and GSE.

A GUE formula is essentially due to Bessis-Itzykson-Zuber 1980:

$$\log \int_{\mathcal{H}_{N,\mathbb{C}}} e^{-\frac{1}{2}N\mathrm{tr}X^2} e^{N\sum_j \frac{t_j}{j}\mathrm{tr}X^j} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Ribbon graph}}} \frac{1}{|\mathrm{Aut}_R\Gamma|} N^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} ,$$

where $\mathcal{H}_{N,\mathbb{C}}$ denotes the space of hermitian matrices of size N, and $e^{-\frac{N}{2}\mathrm{tr}X^2} d\mu(X)$ the normalized probability measure on $\mathcal{H}_{N,\mathbb{C}} = \mathbb{R}^{N^2}$. A *ribbon graph* is a graph with a cyclic order assigned to each vertex.

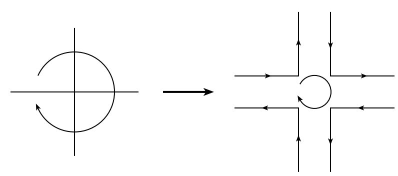

FIGURE 1. A vertex with a cyclic order given to incident half-edges can be placed on an oriented plane. An oriented cross road emerges.

Equivalently, it is a graph Γ that is drawn on a closed oriented surface S such that the complement $S \setminus \Gamma$ is the disjoint union of open disks. We use $f(\Gamma)$ to denote the number of these disks (or *faces*), and $v_j(\Gamma)$ the number of j-valent vertices of Γ. Let

$$v(\Gamma) = \sum_j v_j(\Gamma) \qquad \text{and} \qquad e(\Gamma) = \frac{1}{2} \sum_j j v_j(\Gamma)$$

be the number of vertices and edges of Γ, respectively. Then the genus $g(S)$ of the surface S is given by the formula for the Euler characteristic

$$\chi(S) = 2 - 2g(S) = v(\Gamma) - e(\Gamma) + f(\Gamma).$$

The automorphism group $\mathrm{Aut}_R(\Gamma)$ of a ribbon graph Γ consists of automorphisms of the cell-decomposition of S that is determined by the graph.

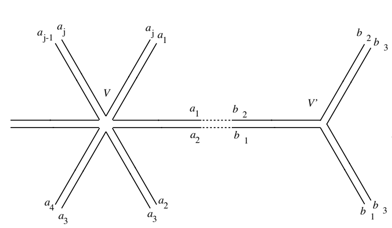

FIGURE 2. A ribbon graph is obtained by connecting cyclically ordered vertices with a ribbon like edge preserving the orientation.

There is a similar formula for GOE, due to many researchers such as Brézin-Itzykson-Parisi-Zuber 1978 and Goulden-Harer-Jackson 2001:

$$\log \int_{\mathcal{H}_{N,\mathbb{R}}} e^{-\frac{1}{4}N\mathrm{tr}X^2} e^{\frac{N}{2}\sum_j \frac{t_j}{j}\mathrm{tr}X^j} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}\Gamma|} N^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} ,$$

where what we call a *Möbius graph* is a graph drawn on a closed surface, orientable or non-orientable, defining a cell-decomposition of the surface. Its automorphism is an automorphism of the cell-decomposition of the surface that is determined by the graph, but this time we allow orientation-reversing automorphisms.

Every compact non-orientable surface without boundary is obtained by removing k disjoint disks from a sphere S^2 and glue k *cross-caps* back into the holes. The number of cross-caps is called the *cross-cap genus* of the non-orientable surface. If S_Γ is non-orientable, then its cross-cap genus k is determined by

$$\chi(S_\Gamma) = 2 - k = v(\Gamma) - e(\Gamma) + f(\Gamma) ,$$

where again by $f(\Gamma)$ we denote the number of disjoint open disks in $S_\Gamma \setminus \Gamma$.

The space $\mathcal{H}_{N,\mathbb{R}}$ of $N \times N$ real symmetric matrices is a real vector space of dimension $N(N+1)/2$, and $d\mu(X)$ is the normalized Lebesgue measure of this space. We note that the coefficients of the integral in GOE case are different from GUE case, reflecting the fact that a dihedral group naturally acts on a vertex of a Möbius graph.

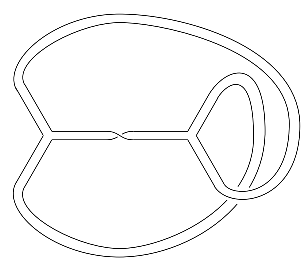

FIGURE 3. A Möbius graph.

A corresponding formula for GSE is only recently discovered in math-ph/0206011:

$$\log \int_{\mathcal{H}_{N,\mathbb{H}}} e^{N\mathrm{tr}X^2} e^{2N\sum_j \frac{t_j}{j}\mathrm{tr}X^j d\mu(X)} = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}\Gamma|} (-2N)^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} ,$$

where $\mathcal{H}_{N,\mathbb{H}}$ is the space of $N \times N$ quaternionic self-adjoint matrices. We pay a particular attention to the negative sign in RHS of the formula and the factor $2N$. This negative sign is responsible for the negative sign appearing in the second formula of our main theorem.

4. Now we ask a question: Why do we obtain graphs drawn on a surface rather than arbitrary graphs?

The cyclic order of a graph in the Feynman diagram expansions we have seen above has its origin in the cyclic invariance of the trace:

$$\mathrm{tr}(X_1X_2X_3\cdots X_{n-1}X_n) = \mathrm{tr}(X_2X_3\cdots X_{n-1}X_nX_1) .$$

For a Möbius graph, we identify two graphs if one is obtained from the other by flipping a vertex and twist every incident edge (see Figure 4). This equivalence has its source in another identity of the trace of a product of matrices

$$\overline{\mathrm{tr}(X_1X_2X_3\cdots X_{n-1}X_n)} = \mathrm{tr}(X_n^*X_{n-1}^*\cdots X_3^*X_2^*X_1^*) .$$

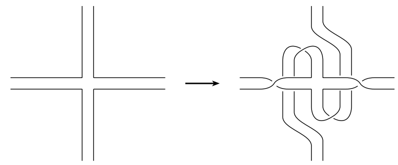

FIGURE 4. Flipping a vertex at the same time twisting all incident edges produces the same Möbius graph.

Therefore, in order to have a ribbon or a Möbius graph, all we need is an algebra with a $*$-operation and a trace. So let us consider a finite-dimensional von Neumann algebra A together with a linear map called *trace*

$$\langle\ \rangle : A \longrightarrow \mathbb{C}$$

satisfying

$$\begin{cases} (ab)^* = b^* a^* \\ \langle 1 \rangle = 1 \\ \langle a^* \rangle = \overline{\langle a \rangle} \\ \langle ab \rangle = \langle ba \rangle \\ \langle aa^* \rangle > 0 \quad \text{for} \quad a \neq 0. \end{cases}$$

We denote by

$$\mathcal{H}_A = \{x \in A \mid x = x^*\}$$

the real vector subspace consisting of self-adjoint elements. We can also consider a *real* von Neumann algebra that is defined over $\mathbb{R}$ as an algebra with a real valued trace. Then we can consider an integral

$$\log \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2 \rangle} e^{\sum \frac{t_j}{j}\langle x^j \rangle} d\mu(x) \ ,$$

where $d\mu(x)$ is a translational invariant Lebesgue measure on A that is also invariant under the linear transformations that preserve the hermitian inner product of A define by

$$\langle x, y \rangle = \langle xy^* \rangle \ .$$

If we apply Feynman diagram expansion to this integral, then the asymptotic series of the integral is a sum over all connected ribbon or Möbius graphs, depending on whether A is a real or a complex algebra.

Question: *Does the above integral (without log) give a τ-function solution to the KP equation in the case of a complex von Neumann algebra, and a solution to the counterpart of the KP equation for the Pfaff Lattice equation of Adler-van Moerbeke in the case of real von Neumann algebra?*

We know that if A is a simple algebra, i.e., a full matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, then the answer is yes. Most likely the KP equation or its counterpart equation implies the simplicity of A.

5. As a typical example for our algebra, let us take the complex group algebra $A = \mathbb{C}[G]$ of a finite group G. We define the $*$-operation by

$$* : \mathbb{C}[G] \ni x = \sum_{\gamma \in G} x^\gamma \cdot \gamma \longmapsto x^* = \sum_{\gamma \in G} \overline{x^\gamma} \cdot \gamma^{-1} \in \mathbb{C}[G] \ .$$

As the trace, we use the character of the regular representation χ_{reg}, by linearly extending to the whole group algebra:

$$\langle\ \rangle = \frac{1}{|G|}\chi_{\mathrm{reg}}\ .$$

Let $\mathcal{H}_{\mathbb{C}[G]}$ denote the real vector subspace of $\mathbb{C}[G]$ consisting of self-adjoint elements. We have a natural Lebesque measure on $\mathcal{H}_{\mathbb{C}[G]} = \mathbb{R}^{|G|}$. Now we have

Theorem 2 (math.QA/0209008)**.**

$$\log \int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right) \exp\left(\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)\right) d\mu(x)$$
$$= \sum_{\substack{\Gamma:\ connected\\ ribbon\ graph}} \frac{1}{|\mathrm{Aut}_R\Gamma|} |G|^{\chi(S_\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_j t_j^{v_j(\Gamma)}\ ,$$

where S_Γ is the oriented surface determined by a ribbon graph Γ.

The application of Feynman diagram expansion gives a sum over connected ribbon graphs whose half-edges are labeled by group elements. The integral is evaluated to be

$$\sum_{\Gamma \text{ ribbon graph}} \frac{1}{|\mathrm{Aut}_R\Gamma|}\mu_\Gamma(G) \prod_j t_j^{v_j(\Gamma)}\ ,$$

where $\mu_\Gamma(G)$ is the number of configurations of assigining group elements to each half-edge of a ribbon graph Γ subject to

Condition 1. If half-edges E_+ and E_- form an edge E of Γ and a group element w is assigned to E_+, then w^{-1} is assigned to E_-;

Condition 2. At every vertex, the product of all group elements assigned to half-edges incident to the vertex according to the cyclic order of the vertex is equal to 1.

The second condition comes from the fact that

$$\langle w_1 w_2 \cdots w_j \rangle = \begin{cases} 1 & w_1 w_2 \cdots w_j = 1 \\ 0 & \text{oherwise} \end{cases}$$

that appears as a j-valent vertex of a graph.

Lemma 3. *The quantity $\mu_\Gamma(G)$ is a topological invariant of a compact oriented surface with a fixed number of marked points (or the faces of its cell-decomposition).*

Proof. This follows from the invariance of $\mu_\Gamma(G)$ under an edge contraction and edge insertion. When an edge that is not a loop is contracted, the configuration of group elements on the new graph still satisfies Conditions 1 and 2. If the edge is inserted back, then we know exactly what group element has to be assigned to each half-edge, due to Condition 2. This proves the lemma. □

Since the graph contribution is a topological invariant for every genus and the number of faces of a ribbon graph, we can use a standard graph for each topology to calculate the number $\mu_\Gamma(G)$, provided that the space of graphs with the same topological type is connected under contraction and expansion. It is known that this is the case indeed. So if we use Figure 5 as our standard graph Γ, then we immediately see that the number of configurations of assignments of group elements on this graph is

$$|G|^{f(\Gamma)-1}|\mathrm{Hom}(\pi_1(S_\Gamma), G)|\ ,$$

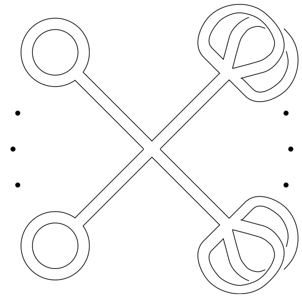

FIGURE 5. A standard graph for a closed oriented surface of genus g with f marked points, or faces. It has $f-1$ tadpoles in the left and g bi-petal flowers in the right.

where the tadpoles of the graph have contribution $|G|^{f(\Gamma)-1}$ in the computation. Since we used the character of the regular representation itself instead of the normalized trace, the power of $|G|$ in Theorem 2 becomes $\chi(S_\Gamma)$.

Note that we have a $*$-algebra isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{\lambda\in\hat{G}} \mathrm{End}(V_\lambda) \ .$$

For $G = \mathfrak{S}_n$, this decomposition is the RSK correspondence. The character of the regular representation also decomposes into the sum of irreducible characters:

$$\chi_{\mathrm{reg}} = \sum_{\lambda\in\hat{G}} (\dim V_\lambda)\chi_\lambda = \sum_{\lambda\in\hat{G}} N_\lambda \, \mathrm{tr}_{V_\lambda} \ ,$$

where $N_\lambda = \dim\lambda$ and χ_λ is its character. Therefore,

$$\begin{aligned}
&\log \int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right) \exp\left(\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)\right) d\mu(x) \\
&= \log \int_{\mathcal{H}_{\mathbb{C}[G]}} \prod_{\lambda\in\hat{G}} \exp\left(-\frac{N_\lambda}{2}\,\mathrm{tr}_{V_\lambda}(x^2)\right) \exp\left(N_\lambda \sum_j \frac{t_j}{j}\mathrm{tr}_{V_\lambda}(x^j)\right) d\mu_\lambda(x) \\
&= \sum_{\lambda\in\hat{G}} \log \int_{\mathcal{H}_{N_\lambda,\mathbb{C}}} \exp\left(-\frac{N_\lambda}{2}\,\mathrm{tr}_{V_\lambda}(x^2)\right) \exp\left(N_\lambda \sum_j \frac{t_j}{j}\mathrm{tr}_{V_\lambda}(x^j)\right) d\mu_\lambda(x) \\
&= \sum_{\substack{\Gamma:\text{ connected}\\ \text{ribbon graph}}} \frac{1}{|\mathrm{Aut}_R\Gamma|} \sum_{\lambda\in\hat{G}} (\dim V_\lambda)^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} \ ,
\end{aligned}$$

where $d\mu_\lambda$ is the normalized Lebesque measure on the space of $N_\lambda \times N_\lambda$ hermitian matrices. Comparing the two expressions of the integral, we obtain the first formula of Theorem 1.

6. For the non-orientable case, we use an integral over the real group algebra $\mathbb{R}[G]$, which is a real $*$-algebra with χ_{reg} as a trace function.

Theorem 4.

$$\begin{aligned}
&\log \int_{\mathcal{H}_{\mathbb{R}[G]}} e^{-\frac{1}{4}\chi_{\mathrm{reg}}(x^2)} e^{\frac{1}{2}\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)} d\mu(x) \\
&\qquad\qquad = \sum_{\substack{\Gamma \textit{ connected}\\ \textit{Möbius graph}}} \frac{1}{|\mathrm{Aut}\Gamma|} |G|^{\chi(S_\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_j t_j^{v_j(\Gamma)} \ ,
\end{aligned}$$

where the integral is taken over the space of self-adjoint elements of $\mathbb{R}[G]$*, and* S_Γ *is the orientable or non-orientable surface determined by a Möbius graph* Γ*.*

Recall that the real group algebra $\mathbb{R}[G]$ decomposes into simple factors according to the three types of irreducible representations $\hat{G}_1$, $\hat{G}_2$, and $\hat{G}_4$. The complex conjugation acts on the set $\hat{G}_2$ without fixed points. Let $\hat{G}_{2+}$ denote a half of $\hat{G}_2$ such that

$$\hat{G}_{2+} \cup \overline{\hat{G}_{2+}} = \hat{G}_2 \ .$$

We have a $*$-algebra isomorphism

$$\mathbb{R}[G] \cong \bigoplus_{\lambda \in \hat{G}_1} M(\dim \lambda, \mathbb{R}) \oplus \bigoplus_{\lambda \in \hat{G}_{2+}} M(\dim \lambda, \mathbb{C}) \oplus \bigoplus_{\lambda \in \hat{G}_4} M(\dim \lambda/2, \mathbb{H}) \ .$$

The algebra decomposition gives a formula for the character of the regular representation on $\mathbb{R}[G]$:

$$\chi_{\text{reg}} = \sum_{\lambda \in \hat{G}_1} (\dim \lambda)\chi_\lambda + \sum_{\lambda \in \hat{G}_{2+}} (\dim \lambda)(\chi_\lambda + \overline{\chi_\lambda}) + \sum_{\lambda \in \hat{G}_4} 2(\dim \lambda) \cdot \text{trace}_{\mathbb{H}} \ ,$$

where in the last term the character is given as the trace of quaternionic $(\dim \lambda)/2 \times (\dim \lambda)/2$ matrices.

The computation of the matrix integral of each factor then establishes the second formula of Theorem 1. Note that the $\hat{G}_2$ component has no contribution in that formula. This is due to the fact that the graphical expansion of a complex hermitian matrix integral contains only oriented ribbon graphs.

7. What one can do for a finite group can be usually extended to a compact group. The counting of $\text{Hom}(\pi_1(S), G)$ is replaced by the volume of the moduli space of flat G-bundles on a Riemann surface S when S is oriented. For example, the case of $G = SU(2)$ should reproduce Witten's result that the volume of the moduli space of flat $SU(2)$-bundles on a Riemann surface of genus g is given by $\zeta(2g-2)$.

What is truly surprising is that if we consider the non-orientable case, then odd values of the Riemann zeta function appear. The geometric object is a *real* algebraic curve $\tilde{S}$, which is obtained by doubling a non-orientable surface S. The representation of $\pi_1(S)$ defines a flat G-bundle on S, which lifts to $\tilde{S}$. The cross-cap genus $k(S)$ of S and the genus $g(\tilde{S})$ of the real algebraic curve $\tilde{S}$ are related by

$$g(\tilde{S}) = k(S) - 1 \ .$$

Since a complex irreducible representation of $SU(2)$ is real if the dimension is odd and quaternionic if the dimension is even, we expect that the volume of the moduli space of flat $SU(2)$-bundles with real structure on a real algebraic curve of genus g is given by

$$\sum_{m=1}^{\infty} (2m-1)^{\chi(S)} + \sum_{m=1}^{\infty} (-2m)^{\chi(S)} = (1 - (1 + (-1)^g)2^{1-g})\zeta(g-1) \ .$$

For both orientable and non-orientable cases, the formulas do not make sense for small genera. Of course our integration method does not work for $L^2(G)$ in a straightforward way, but Peter-Weyl theorem provides a definition of the integral and the decomposition of the regular representation. These subjects will be reported elsewhere.

As a conclusion, I'd like to mention two things. First, integration over a von Neumann algebra should be further investigated. We have considered only a few simple cases in this talk, yet we have encountered some very interesting results.

Secondly, we have seen that consideration of non-orientable surfaces, or real algebraic curves, is of extreme importance. The second formula of Theorem 1 is an evidence.

I would be very pleased if this talk convinced you how wonderful the ideas from matrix integrals are in such remote areas as finite group theory and geometry of surfaces.

Department of Mathematics, University of California, Davis, CA 95616–8633
E-mail address: mulase@math.ucdavis.edu