VOLUME OF REPRESENTATION VARIETIES

MOTOHICO MULASE\(^1\) AND MICHAEL PENKAVA\(^2\)

Abstract. We introduce the notion of volume of the representation variety of a finitely presented discrete group in a compact Lie group using the push-forward measure associated to a map defined by a presentation of the discrete group. We show that the volume thus defined is invariant under the Andrews-Curtis moves of the generators and relators of the discrete group, and moreover, that it is actually independent of the choice of presentation if the difference of the number of generators and the number of relators remains the same. We then calculate the volume of the representation variety of a surface group in an arbitrary compact Lie group using the classical technique of Frobenius and Schur on finite groups. Our formulas recover the results of Witten and Liu on the symplectic volume and the Reidemeister torsion of the moduli space of flat \( G \)-connections on a surface up to a constant factor when the Lie group \( G \) is semisimple.

Contents

0. Introduction and the Main Results 1
1. Presentation of a Group 4
2. Volume of Representation Varieties 6
3. Representation Varieties of Orientable Surface Groups 13
4. Representation Varieties of Non-orientable Surface Groups 16
5. Harmonic Analysis and Irreducible Representations of a Compact Lie Group 20
6. Representation Varieties and Cell-Decompositions of a Surface 23
References 26

0. Introduction and the Main Results

Let \( \Pi \) be a finitely presented discrete group generated by \( k \) elements with \( r \) relations:

\[
(0.1) \quad \Pi = \langle a_1, \ldots, a_k \mid q_1, \ldots, q_r \rangle .
\]

The space \( \text{Hom}(\Pi, G) \) of homomorphisms from \( \Pi \) into a compact Lie group \( G \) is called the representation variety of \( \Pi \) in \( G \). Representation varieties of the fundamental group of a manifold appear in various places in low-dimensional geometry and topology (see for example, [8, 14, 23, 24, 43]). To the presentation (0.1) we associate a presentation map

\[
q : G^k \ni x = (x_1, \ldots, x_k) \mapsto (q_1(x), \ldots, q_r(x)) \in G^r ,
\]

which is a real analytic map. By definition, we have a canonical identification

\[
(0.3) \quad \text{Hom}(\Pi, G) = q^{-1}(1, \ldots, 1)
\]

Date: December 1, 2002.

\(^1\)Research supported by NSF grant DMS-9971371 and the University of California, Davis.
\(^2\)Research supported by NSF grant DMS-0200669 and the University of Wisconsin, Eau Claire.
that gives a realization of the representation variety as a real analytic subvariety of $G^k$. If
the relators are given without redundancy, then its dimension is expected to be
\begin{equation}
\dim \text{Hom}(\Pi, G) = (k - r) \dim G.
\end{equation}

The purpose of this paper is to give a definition of the volume $|\text{Hom}(\Pi, G)|$ of the rep-
resentation variety and to study some of its properties. Since we use the push-forward
measure via the presentation map (0.2) to define the volume, the central question is to
determine whether it is an invariant of the group. In this article we give an affirmative
answer to this question. The main results of the paper are the following.

**Theorem 0.1.** The volume is invariant under the Andrews-Curtis moves.

Because of the dimension formula of (0.4), it is necessary that the difference $k - r$ of the
numbers of generators and relators remains invariant for different presentations of the group
$\Pi$ to give the same volume $|\text{Hom}(\Pi, G)|$. Surprisingly, the condition is sufficient.

**Theorem 0.2.** The volume is independent of the choice of presentation if the difference
between the number of generators and the number of relators is the same.

The volume of the representation variety of the fundamental group of a closed surface is a
topological invariant and actually expressible in terms of irreducible representations of the
compact Lie group $G$.

**Theorem 0.3.** Let $S$ be a closed orientable surface of genus $g \geq 2$. Then the volume of the
representation variety can be computed by the formula
\begin{equation}
\sum_{\lambda \in \hat{G}} (\dim \lambda) \chi(S) = |G|^{\chi(S)} |\text{Hom}(\pi_1(S), G)|
\end{equation}
if the sum of the LHS converges, where $\hat{G}$ denotes the set of isomorphism classes of complex
irreducible representations of $G$, $\dim \lambda$ is the complex dimension of the irreducible rep-
resentation $\lambda$, and $\chi(S) = 2 - 2g$ is the Euler characteristic of $S$. Since $G$ is compact, $\hat{G}$ is a
countable set.

For a non-orientable surface $S$, the formula for the volume involves more detailed infor-
mation on the irreducible representations of $G$. Using the Frobenius-Schur indicator of
irreducible characters [12], we decompose the set of complex irreducible representations $\hat{G}$
into the union of three disjoint subsets, corresponding to real, complex, and quaternionic
irreducible representations:
\begin{equation}
\hat{G}_1 = \left\{ \lambda \in \hat{G} \mid \frac{1}{|\hat{G}|} \int_G \chi_\lambda(w^2) dw = 1 \right\};
\end{equation}
\begin{equation}
\hat{G}_2 = \left\{ \lambda \in \hat{G} \mid \frac{1}{|\hat{G}|} \int_G \chi_\lambda(w^2) dw = 0 \right\};
\end{equation}
\begin{equation}
\hat{G}_4 = \left\{ \lambda \in \hat{G} \mid \frac{1}{|\hat{G}|} \int_G \chi_\lambda(w^2) dw = -1 \right\},
\end{equation}
where $\chi_\lambda$ is the irreducible character of $\lambda \in \hat{G}$. Now we have

**Theorem 0.4.** Let $S$ a closed non-orientable surface. Then we have
\begin{equation}
\sum_{\lambda \in \hat{G}_1} (\dim \lambda) \chi(S) + \sum_{\lambda \in \hat{G}_4} (-\dim \lambda) \chi(S) = |G|^{\chi(S)} |\text{Hom}(\pi_1(S), G)|
if the sum of the LHS is absolutely convergent.

The formulas (0.5) and (0.7) for a finite group, in which case the closed surface can be anything, are due to Mednykh [29] and Frobenius-Schur [12], respectively. We refer to the excellent review article [21] for the history of related topics. For a finite group $G$ and $S = S^2$, (0.5) reduces to the classical formula for the order of the group

$$|G| = \sum_{\lambda \in \hat{G}} (\dim \lambda)^2.$$  

For a compact connected semisimple Lie group $G$, the formulas agree with the results of Witten [52, 54] and Liu [25, 26, 27] up to a constant factor. They calculated the natural symplectic volume and the Reidemeister torsion of the moduli space of flat $G$-connections on a surface $S$ using techniques from theoretical physics and analysis of heat kernels. The idea of using Chern-Simons gauge theory to obtain (0.5) for a finite group was carried out in [10]. The representation variety has algebro-geometric interest as the moduli space of holomorphic vector bundles on a compact Riemann surface, and through this interpretation it has deep connections with conformal field theory and topological quantum field theory. Let $S$ be a compact Riemann surface with a complex structure. Narasimhan-Seshadri theory [37, 38] shows that the moduli space of stable holomorphic $G_C$ bundles over $S$ as a real analytic variety can be identified with

$$\mathcal{M}(S, G_C) = \frac{\text{Hom}(\pi_1(S), G)}{G/Z(G)},$$

where $Z(G)$ is the center of $G$ and the $G$-action on $\text{Hom}(\pi_1(S), G)$ is through conjugation. The first Chern class of the determinant line bundle over the moduli space gives an integral symplectic form $\omega$ on the moduli space. Using this 2-form Witten [52] computed the natural symplectic volume of $\mathcal{M}(S, G_C)$. His formula and ours differ by a factor of $(2\pi)^{\dim G}$, which is due to the normalization of the integral cohomology class $[\omega]$ on the moduli space, but the factor does not come from the choice of the Haar measure on the group $G$. From the algebro-geometric point of view, certainly the quotient of the natural conjugate action of the representation variety is a desirable object of study. However, our formulas (0.5) and (0.7) suggest that the volume of the whole representation variety measured in comparison with the volume of the space $G^{1-{\chi(S)}}$ may be a more natural object. The heat kernel method is necessary to connect the volume of the representation variety of a surface group we discuss in this paper to the symplectic volume or the Reidemeister torsion of the moduli space of flat connections on a closed surface. Our point is that if we consider the natural quantity $|G|^{\chi(S)-1}\text{Hom}(\pi_1(S), G)$, which is independent of the choice of the measure on $G$, then we can compute its value by exactly the same classical method of Frobenius [11] and Frobenius-Schur [12] for finite groups.

The paper is organized as follows. In Section 1 we recall that a group $\Pi$ is completely determined by the hom functor $\text{Hom}(\Pi, \bullet)$. This implies that two presentations define the same group if and only if what we call the presentation functors are naturally isomorphic. With this preparation we define the volume of the representation variety $\text{Hom}(\Pi, G)$ in Section 2 using a presentation of $\Pi$. We then prove that the volume is actually independent of the presentation, utilizing the behavior of the volume under the Andrews-Curtis moves. Then in Sections 3 and 4 we calculate the volume of the representation variety of a surface group, the orientable case first followed by the non-orientable case. Throughout the paper except for Section 1 we need non-commutative harmonic analysis of distributions on a
compact Lie group and character theory. These techniques are summarized in Section 5. The final section is devoted to presenting a heuristic argument that relates the computation of the volume of this paper to the non-commutative matrix integral studied in [35, 36].

Acknowledgement. The authors thank Bill Goldman for drawing their attention to important references on the subject of this article. They also thank Greg Kuperberg, Abby Thompson and Bill Thurston for stimulating discussions on representation varieties and Andrews-Curtis moves.

1. Presentation of a Group

Consider a set of alphabets \( \{a_1, \ldots, a_k\} \) consisting of \( k \) letters. A word \( W(a) \) of length \( \nu \) is the expression of the form

\[
W(a) = a_{w(1)}^{\epsilon(1)} \cdots a_{w(\nu)}^{\epsilon(\nu)},
\]

where the subscript

\[
w : \{1, \ldots, \nu\} \to \{1, \ldots, k\}
\]

indicates the \( \ell \)-th letter \( a_{w(\ell)} \) in the word \( W(a) \), and the exponent \( \epsilon(\ell) = \pm 1 \) represents either the letter \( a_{w(\ell)} \) or its inverse \( a_{w(\ell)}^{-1} \). Instead of the commonly used expression \( a_i^{\nu} \), we always use

\[
\underbrace{a_1 \cdots a_1}_{\nu},
\]

which is a word of length \( \nu \). Throughout this article, every word is assumed to be reduced, namely, no expressions like \( a_i a_i^{-1} \) and \( a_i^{-1} a_i \) appear in a word.

For every given group \( G \), a word \( W(a) \) of letters \( a_1, \ldots, a_k \) (that are unrelated with \( G \)) defines a map

\[
W : G^k \ni x = (x_1, \ldots, x_k) \mapsto W(x) = x_{w(1)}^{\epsilon(1)} \cdots x_{w(\nu)}^{\epsilon(\nu)} \in G.
\]

The map \( W \) is the composition of two maps

\[
G^k \ni (x_1, \ldots, x_k) \xrightarrow{\iota_W} (x_{w(1)}, \ldots, x_{w(\nu)}) \xrightarrow{m} x_{w(1)}^{\epsilon(1)} \cdots x_{w(\nu)}^{\epsilon(\nu)},
\]

where \( \iota_W : G^k \to G^\nu \) represents the shape of the word \( W(a) \), and

\[
m : G^\nu \ni (y_1, \ldots, y_\nu) \mapsto y_1 \cdots y_\nu \in G
\]

is the multiplication map. If the word \( W(a) \) does not contain the letter \( a_i \), then \( x_i \) is sent to \( 1 \in G \) by the map \( \iota_W \). In particular, if \( W = \emptyset \) is the empty word, then \( \iota_\emptyset \) is the trivial map that sends everything in \( G^k \) to \( \{1\} = G^0 \).

Let us consider a finitely presented discrete group

\[
\Pi = \langle a_1, \ldots, a_k \mid q_1, \ldots, q_r \rangle
\]
defined by \( k \) generators and \( r \) relations \( q_1 = \cdots = q_r = 1 \). Every relator \( q_j \) is a reduced word, and we always assume that none of the relators is the empty word. The set of relators defines the presentation map

\[
q : G^k \ni x = (x_1, \ldots, x_k) \mapsto (q_1(x), \ldots, q_r(x)) \in G^r.
\]

We give a group structure to the target space \( G^r \) as the product group of \( r \) copies of \( G \). The identity element of \( G^r \) is \( 1 = (1, \ldots, 1) \). The representation space of \( \Pi \) in \( G \) is the set of group homomorphisms from \( \Pi \) to \( G \) and is naturally identified with

\[
\text{Hom}(\Pi, G) = q^{-1}(1).
\]

Let \( \mathcal{G} \) denote the category of groups with group homomorphisms as morphisms among the objects. A finite presentation (1.6) defines a covariant functor from \( \mathcal{G} \) to the category \( \mathcal{S} \) of sets, associating \( q^{-1}(1) \) to a group \( G \). Let us call this functor the presentation functor. The covariance of the presentation functor is a consequence of the commutativity of the diagram

\[
\begin{array}{ccc}
H^k & \xrightarrow{q} & H^r \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
G^k & \xrightarrow{q} & G^r
\end{array}
\]

for every homomorphism \( \gamma : H \to G \), where the vertical arrows are the natural maps induced by \( \gamma \). This functor is representable and the group \( \Pi \) is the representing object in \( \mathcal{G} \) [28]. Thus the presentation functor, whose representation is the hom functor \( \text{Hom}(\Pi, \cdot) \), is determined by the group \( \Pi \) alone and is independent of the choice of its presentation. Let \( \mu : \Pi \to \Pi' \) be a group homomorphism from \( \Pi \) to \( \Pi' \). The pull-back map \( \mu^* \) defines the natural morphism

\[
\mu^* : \text{Hom}(\Pi', \cdot) \longrightarrow \text{Hom}(\Pi, \cdot)
\]

between the hom functors.

Notice that the functor from \( \mathcal{G} \) to the functor category \( \mathcal{F}(\mathcal{G}, \mathcal{S}) \) that associates

\[
\mathcal{G} \ni \Pi \longmapsto \text{Hom}(\Pi, \cdot) \in \mathcal{F}(\mathcal{G}, \mathcal{S})
\]

is a faithful contravariant functor. Indeed, if there is a natural isomorphism

\[
A : \text{Hom}(\Pi, \cdot) \xrightarrow{\sim} \text{Hom}(\Pi', \cdot),
\]

then by applying this functor to \( \Pi' \in \mathcal{G} \), we obtain a homomorphism \( \mu : \Pi \to \Pi' \) that corresponds to the identity automorphism \( 1_{\Pi'} \in \text{Hom}(\Pi', \Pi') \). Similarly, \( 1_{\Pi} \in \text{Hom}(\Pi, \Pi) \) corresponds to \( \xi : \Pi' \to \Pi \). Because of the naturalness condition, for the homomorphism \( \mu \) we have the commutative diagram

\[
\begin{array}{ccc}
\Pi & \xrightarrow{1_{\Pi}} & \Pi \\
\downarrow{\mu} & & \downarrow{\mu} \\
\Pi' & \xrightarrow{1_{\Pi'}} & \Pi'
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}(\Pi, \Pi) & \xrightarrow{A_{\Pi}} & \text{Hom}(\Pi', \Pi) \\
\downarrow & & \downarrow \\
\text{Hom}(\Pi, \Pi') & \xrightarrow{A_{\Pi'}} & \text{Hom}(\Pi', \Pi')
\end{array}
\]

\[
\begin{array}{ccc}
\Pi & \xleftarrow{1_{\Pi}} & \Pi \\
\downarrow{\mu} & & \downarrow{\mu} \\
\Pi' & \xleftarrow{1_{\Pi'}} & \Pi'
\end{array}
\]
that relates 
\[ 1_{\Pi} \longrightarrow \xi \]
\[ \mu \longrightarrow 1_{\Pi'} \]
In particular, we have \( \mu \circ \xi = 1_{\Pi'} \), hence \( \Pi \cong \Pi' \). As a consequence of this faithfulness, we note that if two presentation functors are naturally isomorphic, then the two presented groups are actually isomorphic.

The above consideration suggests that we can use a particular presentation of a group \( \Pi \) to define the volume of the representation space \( \text{Hom}(\Pi, G) \), which will be an invariant of the group \( \Pi \), independent of its presentation. However, to carry out the calculation of the volume \( |\text{Hom}(\Pi, G)| \), we need not only a presentation of \( \Pi \), but also a more specific natural isomorphism between presentation functors. Notice that two presentations

\begin{equation}
\Pi_1 = \langle a_1, \ldots, a_k \mid q_1, \ldots, q_r \rangle \quad \text{and} \quad \Pi_2 = \langle b_1, \ldots, b_k \mid s_1, \ldots, s_r \rangle
\end{equation}

define the same group if and only if for every \( a_i \) there is a word \( a_i(b) \) of the generators \( b_1, \ldots, b_k \) and for every \( b_j \) there is a word \( b_j(a) \) of \( a_1, \ldots, a_k \) such that for any group \( G \in \mathcal{G} \), the maps \( a \) and \( b \) associated to these words are bijective and satisfy the commutativity

\begin{equation}
G^k \overset{a}{\longrightarrow} G^k \overset{q}{\longrightarrow} G^r
\end{equation}

\begin{equation}
G^k \overset{b}{\longrightarrow} G^k \overset{s}{\longrightarrow} G^r
\end{equation}

and

\begin{equation}
q^{-1}(1) = b^{-1}(s^{-1}(1)), \quad s^{-1}(1) = a^{-1}(q^{-1}(1)).
\end{equation}

In Section 2 we use this specific form of the natural isomorphism of the presentation functors to show the independence of the volume of the representation variety on the choice of presentation.

2. Volume of Representation Varieties

From now on we restrict our attention to the category of compact real analytic Lie groups. The representation space \( \text{Hom}(\Pi, G) \) of

\[ \Pi = \langle a_1, \ldots, a_k \mid q_1, \ldots, q_r \rangle \]

in a compact Lie group \( G \) is called the representation variety. Unless otherwise stated, we do not assume connectivity or semisimplicity for the compact Lie group \( G \). Therefore, it can be in particular a finite group. The natural identification \( (1.8) \) makes \( \text{Hom}(\Pi, G) \) a real analytic subvariety of \( G^r \). In this section we define the volume of the representation variety using techniques from non-commutative harmonic analysis on compact Lie groups. Especially we use theory of distributions on a compact Lie group and character theory freely in Sections 2, 3 and 4. We refer to the summary presented in Section 5 for terminologies, definitions and properties of distributions necessary for our investigation.

We denote by \( dx \) a left and right invariant measure on \( G \). We do not normalize the invariant measure, and call the quantity

\[ |G| = \int_G dx \]
the volume of the group $G$. The invariant measure on the product group $G^r$ is the product measure $dw_1 \cdots dw_r$. The $\delta$-function on $G^r$ is defined by
\begin{equation}
\delta_r(w_1, \ldots, w_r) = \delta(w_1) \cdots \delta(w_r).
\end{equation}
For the presentation map
\[ q : G^k \rightarrow G^r \]
of (1.7), we associate the volume distribution $f_q(w)$ on $G^r$ by
\begin{equation}
f_q(w) = |q^{-1}(w)| = \int_{G^k} \delta_r(q \cdot w^{-1})dx_1 \cdots dx_k,
\end{equation}
where
\[ q \cdot w^{-1} = (q_1(x)w_1^{-1}, \ldots, q_r(x)w_r^{-1}) \in G^r \]
and $dx_1 \cdots dx_k$ is the product measure on $G^k$. The volume distribution is a Schwartz distribution on the group $G^r$ characterized by the continuous linear functional associated with the push-forward measure
\begin{equation}
g(w) \mapsto \int_{G^r} f_q(w)g(w)dw_1 \cdots dw_r = \int_{G^k} g(q(x_1, \ldots, x_k))dx_1 \cdots dx_k
\end{equation}
for every $g(w) \in C^\infty(G^r)$. Sard’s theorem tells us that the set of critical values of $q$ has measure 0 in the target space with respect to the Lebesgue measure of $G$ as a real analytic manifold, and hence also with respect to the Haar measure $dx$. If the distribution $f_q(w)$ is regular at $w = 1$ (see Section 5), then we define the volume of the representation variety by
\begin{equation}
|\text{Hom}(\Pi, G)| = f_q(1).
\end{equation}
The total volume of $G^k = q^{-1}(G^r)$ with respect to the volume distribution is
\[ \int_{G^r} f_q(w)dw_1 \cdots dw_r = \int_{G^k} \left( \int_{G^r} \delta_r(q \cdot w^{-1})dx_1 \cdots dx_r \right)dx_1 \cdots dx_k = |G^k|.
\]
Thus our definition (2.4) agrees with Fubini’s volume if the presentation map $q$ is a fiber bundle. Since evaluation of a distribution at a point does not usually make sense, we caution that the volume is defined only when $f_q(1)$ is regular. This imposes a strong condition on the presentation of $\Pi$.

**Proposition 2.1.** If the presentation map $q : G^k \rightarrow G^r$ has $1 \in G^r$ as a regular value, then the volume distribution $f_q(w)$ is also regular at the identity element.

**Proof.** The condition implies that the differential $dq$ of the presentation map $q$ has the maximal rank at every point of $q^{-1}(1)$, and that $q^{-1}(1)$ is a non-singular submanifold of $G^k$. Since $G$ is compact, there is an open neighborhood $U$ of $1 \in G^r$ such that
\begin{equation}
\phi : F \times U \rightarrow q^{-1}(U)
\end{equation}
as a $C^\infty$ manifold. Here $F = \text{Hom}(\Pi, G) = q^{-1}(1)$ is a $C^\infty$ manifold and the presentation map restricted to $q^{-1}(U)$ is the projection $F \times U \rightarrow U$. Let $\omega$ be the invariant volume form on $G$ corresponding to the invariant measure $dx$. Denote by $p_j : G^k \rightarrow G$ the projection to the $j$-th component. Then
\begin{equation}
\omega^k \equiv p_1^*\omega \wedge \cdots \wedge p_k^*\omega
\end{equation}
Corresponds to the product measure $dx_1 \cdots dx_k$ on $G^k$. Similarly, we denote by $\omega^r$ the invariant volume form on the product group $G^r$ corresponding to $dw_1 \cdots dw_r$. Notice that the volume form $\phi^*\omega^k$ on $F \times U$ is a $C^\infty$ form. Therefore, there is a $C^\infty$ function $h(w)$ on
such that the integration of $\phi^*\omega^k$ along fiber associated with the fibration $F \times U \to U$ is given by

$$\int_{F \times \{w\}} \phi^*\omega^k = h(w) \omega^r.$$ 

Since the volume distribution $f_q(w)$ is the push-forward measure, we have the equality

$$(f_q|U)(w) = h(w)$$

as a distribution. Therefore, the localization $f_q|U$ of the volume distribution $f_q(w)$ to $U$ is a smooth function. □

**Example 2.1.** Consider a presentation

$$\Pi = \langle a, b \mid ab^{-1} = 1 \rangle .$$

The corresponding presentation map is

$$q: G \times G \ni (x, y) \longmapsto xy^{-1} \in G ,$$

hence $q^{-1}(1)$ is the diagonal of $G \times G$. The volume of the diagonal is calculated by

$$|\text{Hom}(\Pi, G)| = \int_{G^2} \delta(xy^{-1})dxdy = |G| .$$

Notice that the same group is presented by $\langle a \rangle$. With this presentation, there is no relation, and the presentation map is the trivial map

$$G \longrightarrow \{1\} = G^0 .$$

The delta function on the trivial group is the constant $1 \in \mathbb{C}$, hence

$$|\text{Hom}(\langle a \rangle, G)| = \int_G dx = |G| .$$

For a compact Lie group $G$, we require that the maps $a$ and $b$ of the natural isomorphism of (1.11) and (1.12) are real analytic automorphisms of the real analytic manifold $G^k$.

**Lemma 2.2.** Suppose

$$b: G^k \ni x = (x_1, \ldots, x_k) \longmapsto (b_1(x), \ldots, b_k(x)) = b(x) \in G^k$$

is an analytic automorphism of the real analytic manifold $G^k$ given by $k$ words $b_1, \ldots, b_k$ of $x_1, \ldots, x_k \in G$ such that its inverse is also given by $k$ words in a similar way. Then the volume form $\omega^k$ of $G^k$ is invariant under the automorphism $b$ up to a sign:

$$b^*\omega^k = \pm \omega^k .$$

**Proof.** Since the automorphism $b$ maps $(1, \ldots, 1) \in G^k$ to itself, the differential $db$ at $(1, \ldots, 1) \in G^k$ can be described in terms of the Lie algebra $\mathfrak{g}$ of $G$. Noticing that a word is a product of $\pm 1$ powers of $x_j$’s, the differential

$$db: T_{(1, \ldots, 1)}G^k = \mathfrak{g}^\oplus k \cong \mathfrak{g}^\oplus k = T_{(1, \ldots, 1)}G^k$$

is given by an invertible $k \times k$ matrix of integer entries $N \in GL(k, \mathbb{Z})$. Since $\det N = \pm 1$, the $G$-invariant volume form on $\mathfrak{g}^\oplus k$ is preserved by $db$ up to the sign $(\det N)^{k\dim G}$.

Now notice that

$$b^*\omega^k = b_1^*\omega \wedge \cdots \wedge b_k^*\omega$$

and

$$\omega^k = p_1^*\omega \wedge \cdots \wedge p_k^*\omega ,$$

for a compact Lie group $G$, we require that the maps $a$ and $b$ of the natural isomorphism of (1.11) and (1.12) are real analytic automorphisms of the real analytic manifold $G^k$.
Choose a volume form $\omega_U$ on $G'$ respectively, defined by (2.6). Then the localizations of the volume distributions on $F$ where $F$ is the restriction of (1.5). Here $\nu$ is the length of $b_j$. Since $\omega$ is left and right multiplication invariant, the restriction of $m^*\omega$ to the $\ell$-th factor of $G'$ is $\omega$ itself. To be more precise, it is $p_j^*\omega$. If the $\ell$-th letter of the word $b_i$ is $x_1$, then the restriction of $\nu$ on $G'$ of (1.4) on the first factor of $G^k$ into the $\ell$-th factor of $G'$ is the identity map. Thus $\nu$ for $G^k$ restricted to the first factor of $G'$ is just an integer multiple of $\omega$ (or to be precise $p_j^*\omega$), and the coefficient is the number of times the letter $x_1$ appears in the word $b_i$ plus $(-1)^{\dim G}$ for each appearance of $x_1^{-1}$ in $b_i$. Define $n_{ij}$ as the number of letter $x_j$ in the word $b_i$ minus the number of $x_j^{-1}$ in $b_i$. Then the matrix $N = (n_{ij})$ is the same matrix $N$ that represents the differential $db_i$. Therefore, $N \in GL(k, Z)$. Noticing the identity
$$b^*\omega = b^*\omega \land \cdots \land b^*\omega = \nu \circ m^*\omega \land \cdots \land \nu \circ m^*\omega,$$ we obtain the desired formula
$$b^*\omega = N^*\omega = (\det N)^{\dim G} \cdot \omega.$$

\[ \square \]

A consequence of this lemma is the independence of the volume $\text{Hom}(\Pi, G)$ on the choice of presentation among those with $k$ generators and $r$ relations.

**Theorem 2.3.** Let $G$ be a compact Lie group. Suppose we have two presentations (1.10) of the same group $\Pi$ with the same $k$ and $r$. If the presentation maps $p$ and $q$ have the regular value at $1 \in G'$, then the volume of the representation variety of $\Pi$ in $G$ is independent of the presentation:

\[ f_q(1) = \int_{G^k} \delta_x(q(x_1, \ldots, x_r)) \, dx_1 \cdots dx_r = \int_{G^k} \delta_x(s(y_1, \ldots, y_k)) \, dy_1 \cdots dy_k = f_s(1). \]

**Proof.** As in Proposition 2.1, there is an open neighborhood $U \ni 1$ of $G'$ and $C^\infty$ maps $\phi$ and $\psi$ such that

$\phi : q^{-1}(U) \leftarrow F \times U \xleftarrow{\sim} (sb)^{-1}(U) \xrightarrow{\psi \circ \text{proj}} \sim U$

where $F = q^{-1}(1) = (sb)^{-1}(1) \subset G^k$. Note in particular that the restrictions of $\phi$ and $\psi$ on $F \times \{1\}$ are the identity map. Let $\omega^k$ and $\omega^r$ denote the volume forms on $G^k$ and $G^r$, respectively, defined by (2.6). Then the localizations of the volume distributions $f_q$ and $f_{sb}$ on $U$ (see Section 5) are $C^\infty$ functions and given by the integration along fiber:

$$\int_{F \times \{w\}} \phi^*\omega^k = (f_q|_U)(w) \omega^r,$$  \quad  $$\int_{F \times \{w\}} \psi^*\omega^k = (f_{sb}|_U)(w) \omega^r.$$

Choose a volume form $\omega_F$ on $F$ and write

$$\phi^*\omega^k = h_q(z, w) \omega_F \land \omega^r \quad \text{and} \quad \psi^*\omega^k = h_{sb}(z, w) \omega_F \land \omega^r.$$
using $C^\infty$ functions $h_q$ and $h_{sb}$ on $F \times U$. Then the integration along fiber is given by

$$\left(f_q|_U\right)(w) \omega^r = \int_{F \times \{w\}} \phi^* \omega^k = \left(\int_{F \times \{w\}} h_q(z, w) \omega_F \right) \omega^r,$$

$$\left(f_{sb}|_U\right)(w) \omega^r = \int_{F \times \{w\}} \psi^* \omega^k = \left(\int_{F \times \{w\}} h_{sb}(z, w) \omega_F \right) \omega^r.$$  

Since $h_q(z, 1) = h_{sb}(z, 1)$ as a function on $F \times \{1\}$, we obtain $f_q(1) = f_{sb}(1)$. Thus if we have $b^* \omega^k = \omega^k$, then $f_{sb}(1) = f_s(1)$, and the proof is completed. This invariance of the volume form $\omega^k$ under the automorphism $b$ is assured by the previous lemma. The sign $\pm 1$ in Lemma 2.2 is not our concern here because we always use the positive volume form for integration on $G^k$. \hfill \square

To establish the independence of the volume on the choice of presentation with different numbers of generators and relators, we first need its invariance under the Andrews-Curtis moves [1].

**Definition 2.4.** The Andrews-Curtis moves of a presentation (1.6) of a discrete group are the following six operations on the generators and relators:

1. Interchange $q_1$ and $q_j$ for $j = 2, \ldots, r$.
2. Replace $q_1$ with $aq_1a^{-1}$ for an element $a \in \Pi$.
3. Replace $q_1$ with $q_1^{-1}$.
4. Replace $q_1$ with $q_1q_2$.
5. Add another generator $a$ and a relator $q_{r+1} = a$.
6. Delete a generator $a$ and a relator $a$.

**Theorem 2.5.** Let $\Pi = \langle a_1, \ldots, a_k \mid q_1, \ldots, q_r \rangle$ be a finite presentation of a discrete group and $G$ a compact Lie group. If the volume distribution $f_q(w)$ is regular at $w = 1 \in G^r$ and hence the volume of the representation variety $f_q(1) = |\text{Hom}(\Pi, G)|$ is well-defined, then the volume is invariant under the Andrews-Curtis moves.

**Proof.** The invariance under the first three moves follows immediately from (2.1) and (5.8). To show the invariance under (4), we note the following identity of $\delta$-functions:

$$(2.10) \quad \delta_2(x, y) = \delta(x)\delta(y) = \delta(xy)\delta(y).$$

This is because for every $g(x, y) \in C^\infty(G^2)$, we have

$$\int_{G^2} g(x, y)\delta(xy)\delta(y)dxdy = \int_{G} g(y^{-1}, y)\delta(y)dy = g(1, 1) = \int_{G^2} g(x, y)\delta_2(x, y)dxdy.$$

Therefore,

$$\delta(q_1)\delta(q_2) \cdots \delta(q_r) = \delta(q_1q_2)\delta(q_2) \cdots \delta(q_r).$$
The invariance under (5) and (6) comes from the following commutative diagram
\[
\begin{array}{ccc}
G^k \times G & \xrightarrow{(q, id)} & G^r \times G \\
\uparrow i_k & & \uparrow i_r \\
G^k & \xrightarrow{q} & G^r
\end{array}
\]  
(2.11)
where the vertical maps are the inclusions into the first $G^k$ and $G^r$ components, respectively, with $1 \in G$ in the last component. The invariance is a consequence of the identity
\[
\int_{G^k \times G} \delta_r(q) \delta(x) dx_1 \cdots dx_k dx = \int_{G^k} \delta_r(q) dx_1 \cdots dx_k.
\]

□

The Andrews-Curtis moves were introduced in [1] with the well-known conjecture stating that every presentation of the trivial group with the same number of generators and relations can be transformed into the standard presentation by a finite sequence of moves (1)–(6). The conjecture is still open to date. For three-manifold groups, the conjecture is expected to be true.

A three-manifold group $\Pi = \pi_1(M)$ of a closed oriented 3-manifold $M$ always has a balanced presentation, meaning that the number of generators and relations are the same, given by a Heegaard splitting of $M$. If $\Pi$ has a balanced presentation, then the expected dimension of the representation variety $\text{Hom}(\Pi, G)$ in a compact Lie group $G$ is 0, and hence it consists of a finite set. The Casson invariant counts the number of elements of $\text{Hom}(\pi_1(M), SU(2))$ with sign. If indeed $\text{Hom}(\pi_1(M), G)$ is a finite set, then its cardinality $|\text{Hom}(\pi_1(M), G)|$ is a topological invariant of $M$. For example, Kuperberg [24] discussed this quantity for a finite group $G$. But often the representation variety contains higher dimensional exceptional components. If such a situation happens, then the volume distribution becomes singular at the identity element of $G^r$, and our theory does not provide any useful information.

We are now ready to prove that the volume of the representation variety is independent of the choice of presentation, if it is well-defined. More precisely, we have

**Theorem 2.6.** Suppose a group $\Pi$ has two presentations of the form
\[
\langle a_1, \ldots, a_k \mid q_1, \ldots, q_r \rangle
\]
and
\[
\langle b_1, \ldots, b_\ell \mid s_1, \ldots, s_t \rangle
\]
such that
\[
k - r = \ell - t,
\]
(2.14)
or in other words, the expected dimensions of $\text{Hom}(\Pi, G)$ through the two presentations agree. If the volume distributions $f_q$ and $f_s$ are regular at the identity element of the product groups $G^r$ and $G^t$, respectively, then
\[
|\text{Hom}(\Pi, G)| = f_q(1) = f_s(1).
\]
Proof. Without loss of generality, we can assume $k \leq \ell$. Using the Andrews-Curtis moves, we supply the new generators $a_{k+1}, \ldots, a_{\ell}$ and relators
\[ q_{r+1} = a_{k+1}, \ldots, q_{r+\ell-k} = a_{\ell} \]
to the first presentation (2.12) so that we have
\begin{equation}
\Pi = \langle a_1, \ldots, a_{\ell} \mid q_1, \ldots, q_{r+\ell-k} \rangle.
\end{equation}
From (2.14), we know $r+\ell-k = t$. Therefore, the two presentations (2.15) and (2.13) have the same numbers of generators and relators. If $f_q(w)$ is regular at $w = 1 \in G^r$, then the volume distribution of the enlarged presentation (2.15) is also regular at $1 \in G^\ell$. Therefore, from Theorem 2.3, we conclude that
\[ f_q(1) = f_s(1). \]
\[ \square \]

For a one-relation group such as a surface group, character theory of compact Lie groups provides a computational tool. In the next two sections, we calculate the volume distribution for every surface group in terms of irreducible characters of $G$.

Lemma 2.7. Let $\Pi = \langle a_1, \ldots, a_k \mid q \rangle$ be a one-relation group. Then for every compact Lie group $G$, the volume distribution (2.2) is a class distribution on $G$.

Proof. For $y \in G$ and $g(w) \in C^\infty(G)$, we have
\[
\int_G f_q(gwy^{-1})g(w)dw = \int_G f_q(w)g(y^{-1}wy)dw
\]
\[
= \int_{G^k} g(y^{-1}q(x_1, \ldots, x_k)y)dx_1 \cdots dx_k
\]
\[
= \int_{G^k} g(q(y^{-1}x_1y, \ldots, y^{-1}x_ky))dx_1 \cdots dx_k
\]
\[
= \int_{G^k} g(q(x_1, \ldots, x_k))dx_1 \cdots dx_k
\]
\[
= \int_G f_q(w)g(w)dw
\]
because of the left and right invariance of the Haar measure. \[ \square \]
Since the irreducible characters are real analytic functions on $G$ and form an orthonormal basis for the $L^2$ class functions on $G$, we have

Lemma 2.8. The character expansion of the class distribution $f_q(w)$ on $G$ is given by
\begin{equation}
f_q(w) = \frac{1}{|G|} \sum_{\lambda \in \hat{G}} \left( \int_{G^k} \chi_{\lambda}(q(x_1, \ldots, x_k))dx_1 \cdots dx_k \right) \chi_{\lambda}(w).
\end{equation}
Notice that the integral in (2.16) is convergent and well-defined, and the infinite sum is convergent with respect to the strong topology of $\mathcal{D}'(G)$. Thus the regularity question of $f_q(w)$ can be answered by studying the convergence problem of the infinite sum over all irreducible representations. For example, if the sum of (2.16) is convergent with respect to the sup norm on $G$, then $f_q$ is a $C^\infty$ function on $G$ (see Section 5).
3. Representation Varieties of Orientable Surface Groups

In this section we calculate the volume distribution of the surface group \( \pi_1(S) \) in \( G \) and the volume \( |\text{Hom}(\pi_1(S), G)| \) of the representation variety for a closed orientable surface \( S \) and a compact Lie group \( G \). In this and the next section, the statements and the argument of proof are actually almost identical to that of the classical papers by Frobenius [11, 12]. Because of the way we define the volume in the previous section, the same argument simultaneously works for both finite groups and compact Lie groups.

Let \( S \) be a closed orientable surface of genus \( g \geq 1 \). Its fundamental group is given by

\[
\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle ,
\]

where \( [a, b] = aba^{-1}b^{-1} \) is the group commutator of \( a \) and \( b \). The presentation map is a real analytic map

\[
q_g : G^{2g} \ni (x_1, y_1, \ldots, x_g, y_g) \mapsto [x_1, y_1] \cdots [x_g, y_g] \in G .
\]

The representation variety of \( \pi_1(S) \) in \( G \) is a real analytic subvariety \( \text{Hom}(\pi_1(S), G) = q_g^{-1}(1) \) of \( G^{2g} \). For every \( g \geq 1 \), we denote by \( f_g \) the volume distribution

\[
f_g(w) = f_{q_g}(w) = \int_{G^{2g}} \delta([x_1, y_1] \cdots [x_g, y_g]w^{-1}) \, dx_1 \cdots dx_g \, dy_1 \cdots dy_g .
\]

By Lemma 2.7, \( f_g \) is a class distribution. Our purpose is to calculate its character expansion. From the convolution property of the delta function

\[
\delta(q_{g+h}w^{-1}) = \int_G \delta(q_hw^{-1}u^{-1})\delta(uq_g) \, du ,
\]

we have

\[
f_{g+h} = f_g \ast f_h = f_h \ast f_g
\]

for every \( g, h \geq 1 \). More precisely, consider the breakdown of \( q_{g+h} \):

\[
G^{2g} \times G^{2h} \ni (x_1, y_1, \ldots, x_{g+h}, y_{g+h})
\]

\[
\downarrow
\]

\[
G \times G \ni ([x_1, y_1] \cdots [x_g, y_g], [x_{g+1}, y_{g+1}] \cdots [x_{g+h}, y_{g+h}])
\]

\[
\downarrow
\]

\[
G \ni [x_1, y_1] \cdots [x_{g+h}, y_{g+h}]
\]

The distribution \( f_{g+h} \) is the push-forward of the product measure on \( G^{2g+2h} \) via \( q_{g+h} \), which factors into the convolution product according to (3.4). Here we have used the fact that the convolution of two distributions \( f_g \) and \( f_h \) corresponds to the push-forward measure via the multiplication map \( G \times G \rightarrow G \). In particular, \( f_g \) is the \( g \)-th convolution power of \( f_1 \):

\[
f_g = f_1 \ast \cdots \ast f_1 .
\]

Therefore, it suffices to find the character expansion of \( f_1 \). The following formula and its proof is essentially due to Frobenius [11] published in 1896. We translate his argument from a finite group to a compact Lie group using the Dirac delta function and integration over the group.
Proposition 3.1. As a distribution on $G$, we have the following strongly convergent character expansion formula in $D'(G)$:

\[
(3.5) \quad f_1(w) = \int_{G^2} \delta(xy^{-1}y^{-1}w^{-1})dxdy = \sum_{\lambda \in \hat{G}} \frac{|G|}{\dim \lambda} \chi_{\lambda}(w).
\]

Proof. We begin with the convolution formula for the push-forward measure

\[
\delta(xy^{-1}y^{-1}w^{-1}) = \int_G \delta(x(w^{-1})^{-1})\delta(yx^{-1}y^{-1}u^{-1})du.
\]

We use the class distribution of (5.16) and its character expansion (5.17) here to obtain

\[
\eta_{x^{-1}}(u) = \int_G \delta(yx^{-1}y^{-1}u^{-1})dy = \sum_{\lambda \in \hat{G}} \chi_{\lambda}(x^{-1})\chi_{\lambda}(u) = \sum_{\lambda \in \hat{G}} \chi_{\lambda}(x)\chi_{\lambda}(u).
\]

Then we have

\[
f_1(w) = \int_{G^2} \delta(xy^{-1}y^{-1}w^{-1})dxdy
= \int_{G^3} \delta(x(w^{-1})^{-1})\delta(yx^{-1}y^{-1}u^{-1})dudxdy
= \int_{G^2} \delta(x(w^{-1})^{-1})\eta_{x^{-1}}(u)dudx
= \sum_{\lambda \in \hat{G}} \int_{G^2} \delta(x(w^{-1})^{-1})\chi_{\lambda}(x)\chi_{\lambda}(u)dudx
= \sum_{\lambda \in \hat{G}} \int_G \chi_{\lambda}(w^{-1})\chi_{\lambda}(u)du
= \sum_{\lambda \in \hat{G}} \frac{|G|}{\dim \lambda} \chi_{\lambda}(w).
\]

\[\square\]

Taking the $g$-th convolution power of $f_1$, we obtain

Theorem 3.2. For every $g \geq 1$, the character expansion of the volume distribution $f_g = f_{qg}$ is given by

\[
(3.6) \quad f_g(w) = f_{qg}(w) = \sum_{\lambda \in \hat{G}} \left(\frac{|G|}{\dim \lambda}\right)^{2g-1} \chi_{\lambda}(w).
\]

If the sum of the RHS converges with respect to the sup norm on $G$, then the value of $f_g(w)$ gives the volume $|q_g^{-1}(w)|$, and in particular, we have

\[
(3.7) \quad |G|^{\chi(S)-1}|\text{Hom}(\pi_1(S), G)| = \sum_{\lambda \in \hat{G}} (\dim \lambda)^{\chi(S)}.
\]

Conversely, if the RHS of (3.7) is convergent, then the character expansion (3.6) is uniformly and absolutely convergent and the volume distribution is a $C^\infty$ class function.
Remark. Except for the finite group case, the RHS of (3.7) never converges for $g = 0$ and $g = 1$. The Weyl dimension formula gives a lower bound on the genus $g$ for the series to converge. For example, if $G = SU(n)$, then the series (3.7) converges for every $g \geq 2$.

We note that the argument works without any modification for a finite group. If $G$ is a finite group, then there is a canonical choice of the Haar measure: a discrete measure of uniform weight 1 for each element. With this choice, the volume $|G|$ is the order of the group and the delta function is the characteristic function that takes value 1 at the identity element and 0 everywhere else. There is no difference between distributions and differentiable functions in this case, and there is no problem of convergence in the character expansion. Thus $f_g(w)$ of (3.6) is a function with a well-defined value for every $w \in G$. In particular, the expression of $f_g(1)$ for $g \geq 1$ recovers Mednykh’s formula [29] (Theorem 0.3).

For a surface of genus 0, the formula reduces to the classical formula (0.8) and remains true.

For the regularity of the distribution of $f_g(w)$ at $w = 1$, we have the following:

**Proposition 3.3.** Let $S$ be a closed orientable surface of genus $g$ and $G$ a compact connected semisimple Lie group of dimension $n$. If $g \geq n$, then the volume distribution of (3.3) is regular at $w = 1$.

**Proof.** Since $G$ is semisimple, its Lie algebra $\mathfrak{g}$ satisfies
\begin{equation}
D(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}.
\end{equation}
This means every element of $G$ near the identity is expressible as a finite product of commutators. Let us analyze the map
\[ q_1 : G \times G \ni (x, y) \mapsto yxy^{-1}y^{-1} \in G \]

near the identity element. Put $x = e^{tA}$ and $y = e^{sB}$ for some $A, B \in \mathfrak{g}$. Then from the Campbell-Hausdorff formula we have
\[
xyx^{-1}y^{-1} = e^{tA + sB + \frac{t}{2}[A,B] + \cdots} e^{-tA - sB + \frac{t}{2}[A,B] + \cdots} = 1 + ts[A, B] + \text{higher}.
\]

Therefore, for a fixed $y = e^B$, the tangent line at the identity of the image of the one-parameter subgroup $e^{tA}$ via $q_1$ is given by $t[A, B] \in \mathfrak{g}$.

Now let $(A_1, B_1, \ldots, A_g, B_g) \in \mathfrak{g}^{2g}$ be an arbitrary $2g$-tuple of Lie algebra elements. The differential of the map $q_g : G^{2g} \to G$ at $(1, \ldots, 1)$ of $G^{2g}$ is
\begin{equation}
\begin{aligned}
dq_g : \mathfrak{g}^{2g} &\ni (A_1, B_1, \ldots, A_g, B_g) \mapsto [A_1, B_1] + \cdots + [A_g, B_g] \in \mathfrak{g}.
\end{aligned}
\end{equation}

If $g \geq n = \dim G$, then $dq_g$ is surjective because of (3.8). This implies that the map $q_g$ is regular at $(1, \ldots, 1)$.

Let $(a_1, b_1, \ldots, a_g, b_g) \in q_g^{-1}(1)$ be an arbitrary point of the inverse image of 1 via $q_g$. Since the exponential map is surjective, we can write $a_j = e^{X_j}$ and $b_j = e^{Y_j}$ for $j = 1, \ldots, g$. Then again by the Campbell-Hausdorff formula, the differential $dq_g$ at $(a_1, b_1, \ldots, a_g, b_g)$ gives for $(A_1, B_1, \ldots, A_g, B_g) \in \mathfrak{g}^{2g}$ the value
\[
\sum_{j=1}^{g} \left( [X_j, B_j] + [A_j, Y_j] + [A_j, B_j] \right) + \cdots
\]
where we used the fact that $(a_1, b_1, \ldots, a_g, b_g)$ is in the inverse image of 1. This is a surjective map onto $\mathfrak{g}$. Therefore, $q_g$ is locally a fibration in a neighborhood of the identity, and hence the volume distribution $f_g(w)$ is regular at $w = 1$ by Proposition 2.1. \qed
Remark. (1) The semisimplicity condition of $G$ is necessary. Indeed, if $G$ has a center $Z$ of positive dimension, then $Z \times G$ is in $q_1^{-1}(1)$, which has a larger dimension than the expected dimension of the inverse image. Thus the map $q_0$ has critical value at 1, and hence $f_g(w)$ may not be regular at $w = 1$. In particular, the volume distribution is always singular at the identity for any compact abelian group of positive dimension.
(2) Even though $f_g(1)$ is regular, it does not imply that the series of the RHS of (3.7) is convergent.

Example 3.1. Let us consider the case $G = SU(2)$. Since $\hat{G} = N$, we identify $\lambda$ with a positive integer $n$, which is the dimension of $\lambda$. The character expansion (3.6) is uniformly and absolutely convergent for $g \geq 2$, and we have for every closed orientable surface $S$ of $g \geq 2$,

$$\frac{|\text{Hom}(\pi_1(S), SU(2))|}{|SU(2)|^{2g-1}} = \zeta(2g - 2).$$

4. Representation Varieties of Non-orientable Surface Groups

Let us now turn our attention to the case of a non-orientable surface $S$. We recall that every closed non-orientable surface is obtained by removing $k \geq 1$ disjoint disks from a sphere $S^2$ and gluing a cross-cap to each hole. The number $k$ is the cross-cap genus of $S$, and its Euler characteristic is given by $\chi(S) = 2 - k$. Since the fundamental group of a non-orientable surface of cross-cap genus $k$ is given by

$$\pi_1(S) = \langle a_1, \ldots, a_k | a_1^2 \cdots a_k^2 \rangle,$$

we use the presentation map

$$r_k : G^k \ni (x_1, \ldots, x_k) \mapsto x_1^2 \cdots x_k^2 \in G.$$

Our aim is to compute the volume distribution

$$h_k(w) = f_{r_k}(w) = |r_k^{-1}(w)| = \int_{G^k} \delta(r_k(x_1, \ldots, x_k) \cdot w^{-1})dx_1 \cdots dx_k$$

$$= \int_{G^k} \delta(x_1^2 \cdots x_k^2 w^{-1})dx_1 \cdots dx_k.$$

Using the same technique of Section 3, it follows from the convolution property of the $\delta$-function that

$$h_k = \underbrace{h_1 \ast \cdots \ast h_1}_{k\text{-times}}.$$

Proposition 4.1. The character expansion of the class distribution $h_1$ is given by

$$h_1(w) = \int_G \delta(x^2 w^{-1})dx = \sum_{\lambda \in \hat{G}} \phi(\lambda) \chi_\lambda(w),$$

where

$$\phi(\lambda) = \frac{1}{|G|} \int_G \chi_\lambda(x^2)dx$$

is the Frobenius-Schur indicator of [12].
Proof. Let \( h_1(w) = \sum \alpha \lambda \chi_\lambda(w) \) be the expansion in terms of irreducible characters with respect to the strong convergence in \( D(G) \). By definition, we have

\[
a_\lambda = \frac{1}{|G|} \int_G h_1(w) \overline{\chi_\lambda(w)} \, dw
= \frac{1}{|G|} \int_G \int_G \delta(x^{-w} \cdot w) \overline{\chi_\lambda(w)} \, dw \, dx
= \frac{1}{|G|} \int_G \chi_\lambda(x^2) \, dx
= \frac{1}{|G|} \int_G \chi_\lambda(x^2) \, dx.
\]

\( \square \)

Remark. Since the presentation map \( r_1 \) has the same domain and the range, if the space \( r_1^{-1}(1) \) of involutions of \( G \) has positive dimension, then the map \( r_1 : G \ni x \mapsto x^2 \in G \) does not have regular value at \( 1 \in G \). Let \( G \) be a compact connected Lie group of rank \( r \) and \( T \) a maximal torus of \( G \). A compact torus of dimension \( r \) has exactly \( 2^r \) involutions. The conjugacy class of each of these involutions has positive dimension unless the involution is central. Thus the groups \( SO(n) \) and \( SU(n) \) for \( n \geq 3 \) and \( Sp(n) \) for \( n \geq 2 \) all have large space of involutions with positive dimensions. Therefore, the volume \( h_1(1) \) is ill-defined for these groups.

Taking the \( k \)-th convolution power of \( h_1 \), we have

**Theorem 4.2.** Let \( S \) be a closed non-orientable surface of cross-cap genus \( k \geq 1 \) and \( G \) an arbitrary compact Lie group. Then the character expansion of the volume distribution is given by

\[
h_k(w) = |r_k^{-1}(w)| = \sum_{\lambda \in G_1} \left( \frac{|G|}{\dim \lambda} \right)^{k-1} \chi_\lambda(w) - \sum_{\lambda \in G_4} \left( - \frac{|G|}{\dim \lambda} \right)^{k-1} \chi_\lambda(w).
\]

If the character sum of the RHS is absolutely convergent with respect to the sup norm on \( G \), then the value gives the volume \( |r_k^{-1}(w)| \). In particular, we have

\[
|G|^{\chi(S)-1} |\text{Hom}(\pi_1(S), G)| = \sum_{\lambda \in G_1} (\dim \lambda)^\chi(S) + \sum_{\lambda \in G_4} (- \dim \lambda)^\chi(S).
\]

Conversely, if the RHS of \((4.7)\) is absolutely convergent, then \((4.6)\) is uniformly and absolutely convergent and the volume distribution \( h_k \) is a \( C^\infty \) class function.

**Proof.** All we need is to notice the value of the Frobenius-Schur indicator \((0.6)\). \( \square \)

**Example 4.1.** Consider an \( n \)-dimensional torus \( G = T^n \). Since the only real irreducible representation is the trivial representation, \( G_1 = \{ 1 \} \). Let \( S^1 = \{ e^{i\theta} \mid 0 \leq \theta < 2\pi \} \) and choose an invariant measure \( d\theta \) on it. The \( \delta \)-function on the torus is the product of
\(\delta\)-functions on \(S^1\). Thus we have
\[
h_1(w) = \int_{T^n} \delta(2\theta_1 - w_1) \cdots \delta(2\theta_n - w_n) d\theta_1 \cdots d\theta_n
= \left( \int_0^{2\pi} \delta(2\theta) d\theta \right)^n
= \left( \frac{1}{2} \int_0^{4\pi} \delta(\theta) d\theta \right)^n
= 1.
\]
Therefore, \(h_k(w) = |T^n|^{k-1}\), and hence \(|\text{Hom}(\pi_1(S), T^n)| = |T^n|^{1 - \chi(S)}\). The inverse image \(r^{-1}_k(w)\) consists of \((x_1, \ldots, x_k) \in (T^n)^k\) such that
\[
(4.8) \quad x_2^2 = x_{k-1}^{-2} \cdots x_1^{-2} w.
\]
Thus we can choose arbitrary \((x_1, \ldots, x_{k-1}) \in (T^n)^{k-1}\), and for this choice, there are still \(2^n\) solutions of (4.8). Each piece has the volume equal to \(\frac{|T^n|^{k-1}}{2^n}\). Therefore, the total volume is \(|T^n|^{k-1}\), in agreement with our computation.

**Example 4.2.** For \(G = SU(2)\), we have a more interesting relation. We note that \(\hat{G}_1\) consists of odd integers and \(\hat{G}_4\) even integers. Therefore, the series (4.7) is absolutely convergent for \(k \geq 4\). Thus for a closed non-orientable surface of cross-cap genus \(k \geq 4\), we have
\[
\frac{|\text{Hom}(\pi_1(S), SU(2))|}{|SU(2)|^{k-1}} = \sum_{n=1}^{\infty} (2n-1)^{2-k} + (-1)^{2-k} \sum_{n=1}^{\infty} (2n)^{2-k}
= \begin{cases} 
\zeta(k-2) & k \text{ is even} \\
\left(1 - \frac{1}{2^{k-1}}\right) \zeta(k-2) & k \text{ is odd}
\end{cases}
\]
Although the series is not absolutely convergent, we expect that the value for \(k = 3\) is \(\log(2)\).

In the same way as in Proposition 3.3, we have a general regularity condition of the volume distribution \(h_k(w)\) for a compact semisimple Lie group.

**Proposition 4.3.** Let \(G\) be a compact connected semisimple Lie group of dimension \(n\) and \(S\) a closed non-orientable surface of cross-cap genus \(k \geq 1\). Define
\[
m = \left\lfloor \frac{k-1}{2} \right\rfloor = \begin{cases} 
(k-1)/2 & k \text{ is odd} \\
(k-2)/2 & k \text{ is even}
\end{cases}
\]
Then the volume distribution \(h_k(w)\) is regular at \(w = 1\) if \(m \geq n\).

**Proof.** The argument is based on another presentation of the fundamental group containing \(m\) commutator products
\[
\pi_1(S) = \langle a_1, b_1, \ldots, a_m, b_m, c \mid [a_1, b_1] \cdots [a_m, b_m] c^2 \rangle
\]
for odd \(k\) and
\[
\pi_1(S) = \langle a_1, b_1, \ldots, a_m, b_m, c_1, c_2 \mid [a_1, b_1] \cdots [a_m, b_m] c_1^2 c_2^2 \rangle
\]
for even $k$. First we notice that the substitution
\begin{align}
\begin{cases}
\alpha = abc \\
\beta = c^{-1}b^{-1}a^{-1}c^{-1}a^{-1}c \\
\gamma = c^{-1}ac^2
\end{cases}
\end{align}
(4.9)
yields the equality
\begin{equation}
[a, b] c^2 = \alpha^2 \beta^2 \gamma^2.
\end{equation}
(4.10)
The inverse transformation of (4.9) is given by
\begin{align}
\begin{cases}
a = \alpha \beta \gamma \beta^{-1} \alpha^{-1} \gamma^{-1} \beta^{-1} \alpha^{-1} \\
b = \alpha \beta \gamma \alpha \beta \gamma^{-1} \beta^{-1} \gamma^{-1} \beta^{-1} \alpha^{-1} \\
c = \alpha \beta \gamma
\end{cases}
\end{align}
(4.11)
Therefore, the map
\[ h : G^3 \owns (x, y, z) \mapsto (xyz, z^{-1}y^{-1}x^{-1}z^{-1}x^{-1}z, z^{-1}x^2z) = (u, v, w) \in G^3 \]
defined by (4.9) is an analytic automorphism of the manifold $G^3$. Since both $h$ and $h^{-1}$ are given by words, from Lemma 2.2 we conclude
\begin{equation}
h^* \omega^3 = \omega^3.
\end{equation}
(4.12)
Indeed, the integer matrix $N$ representing the differential $dh$ (see proof of Lemma 2.2) is
\[ N = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \]
with $\det N = 1$. Let
\[ s_3 : G^3 \owns (x, y, z) \mapsto [x, y] z^2 \in G \]
be the presentation map with respect to the relator $[a, b] c^2$, and
\[ r_3 : G^3 \owns (u, v, w) \mapsto u^2 v^2 w^2 \in G \]
the presentation map of $\alpha^2 \beta^2 \gamma^2$. Then (4.10) implies that the following diagram commutes:
\[ \begin{array}{ccc}
G^3 & \xrightarrow{s_3} & G \\
\downarrow h & & \downarrow h \\
G^3 & \xrightarrow{r_3} & G \\
\end{array} \]
Together with the invariance of the volume form (4.12), we obtain
\[ f_{s_3}(w) = f_{r_3}(w) = h_3(w) \]
as a distribution on $G$.
We can apply the above procedure based on the transformation (4.9) repeatedly to change the relator
\[ [a_1, b_1] \cdots [a_m, b_m] c^2 \quad \text{to} \quad \alpha_1^2 \beta_1^2 \cdots \alpha_m^2 \beta_m^2 \gamma^2, \]
and
\[ [a_1, b_1] \cdots [a_m, b_m] c_1^2 c_2^2 \quad \text{to} \quad \alpha_1^2 \beta_1^2 \cdots \alpha_m^2 \beta_m^2 \gamma_1^2 \gamma_2^2. \]
At each step the volume form on $G^k$ is kept invariant, and we have the equality of the volume distributions

$$f_{s_k}(w) = h_k(w),$$

where $s_k : G^k \rightarrow G$ is the presentation map associated with the relator containing $m$ commutators. The regularity of $f_{s_k}(w)$ at $w = 1 \in G$ can be established in the same way as in Proposition 3.3. The regularity of $h_k(w)$ follows from (4.13). □

5. Harmonic Analysis and Irreducible Representations of a Compact Lie Group

In this section we briefly review the necessary accounts from the character theory of compact Lie groups and non-commutative harmonic analysis that are used in the main part of this paper. The important results we need are Schur’s orthogonality relations of irreducible characters in terms of the convolution product, definition of the $\delta$-function on a compact Lie group, and expansion of a class distribution in terms of irreducible characters. We refer to [6, 13, 49, 50, 51] for more detail.

Let $G$ be a compact Lie group. We choose a left and right invariant, but not necessarily normalized, measure $dx$ (Haar measure) on $G$ that satisfies

$$\int_G g(x)dx = \int_G g(1x)dx = \int_G g(xy)dx = \int_G g(x^{-1})dx,$$

where $g(x)$ is a smooth function on $G$ and $y \in G$. The space of distributions $\mathcal{D}'(G)$ is the topological linear space consisting of all continuous linear functions on $C^\infty(G)$ with respect to the Fréchet topology of $C^\infty(G)$ [13, 51]. The left regular representation of $G$ on $\mathcal{D}'(G)$ is defined by

$$\rho_L(w) : \mathcal{D}'(G) \ni f(x) \longmapsto f(w^{-1}x) \in \mathcal{D}'(G),$$

where the distribution $f(w^{-1}x)$ is characterized by

$$\int_G f(w^{-1}x)g(x)dx = \int_G f(x)g(wx)dx$$

for every $g(x) \in C^\infty(G)$ using the invariance (5.1) of $dx$. Similarly, the right regular representation is defined by

$$\rho_R(w) : \mathcal{D}'(G) \ni f(x) \longmapsto f(xw) \in \mathcal{D}'(G).$$

Reflecting the group structure of the compact space $G$, $\mathcal{D}'(G)$ has the structure of an associative algebra over $\mathbb{C}$ whose multiplication is the convolution product of distributions $f_1$ and $f_2$ defined by

$$(f_1 \ast f_2)(x) = \int_G f_1(xw^{-1})f_2(w)dw.$$

A distribution $f(x) \in \mathcal{D}'(G)$ is a class distribution if

$$\rho_L(w^{-1}) \cdot f(x) = \rho_R(w) \cdot f(x),$$
or more conveniently, when \( f(w^{-1}xw) = f(x) \) as a distribution for every \( w \in G \). Class distributions form the center of the convolution algebra \( \mathcal{D}'(G) \) because of the invariance of the Haar measure. Indeed, for every smooth function \( g(x) \in C^\infty(G) \), the condition
\[
(f \ast g)(x) = \int_G f(xw^{-1})g(w)\,dw = \int_G g(xu^{-1})f(u)\,du = (g \ast f)(x)
\]
for \( f(x) \in \mathcal{D}'(G) \) is equivalent to (5.6) because
\[
\int_G g(xu^{-1})f(u)\,du = \int_G f(w^{-1}x)g(w)\,dw.
\]
The Dirac \( \delta \)-function on \( G \) is defined by
\[
\int_G \delta(x)g(x)\,dx = g(1), \quad g(x) \in C^\infty(G).
\]
This definition, together with the left and right invariance of the Haar measure \( dx \), implies the following formulas for an arbitrary \( y \in G \):
\[
\int_G \delta(xy^{-1})f(x)\,dx = f(y), \quad f(x) \in \mathcal{D}'(G)
\]
\[
\delta(yxy^{-1}) = \delta(x) = \delta(x^{-1})
\]
\[
\delta(xy) = \delta(yx).
\]
Therefore, the \( \delta \)-function is a class distribution and serves as the identity element of the convolution algebra \( (\mathcal{D}'(G), \ast) \). In particular, we have \( \delta \ast \delta = \delta \).

The Hermitian inner product of \( L^2 \) functions \( f_1 \) and \( f_2 \) on \( G \) is given by
\[
\langle f_1, f_2 \rangle = \frac{1}{|G|} \int_G f_1(x)\overline{f_2(x)}\,dx.
\]
Peter-Weyl theory provides a Hilbert basis for \( L^2(G) \). The \( L^2 \) class functions on \( G \) form a Hilbert space with a Hilbert basis
\[
\{ \chi_\lambda \mid \lambda \in \hat{G} \},
\]
where \( \hat{G} \) is the set of equivalence classes of complex irreducible representations of \( G \) and \( \chi_\lambda \) the character of \( \lambda \in \hat{G} \). The irreducible characters satisfy Schur’s orthogonality relations
\[
\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu}
\]
and
\[
\chi_\lambda \ast \chi_\mu = \frac{|G|}{\dim \lambda} \delta_{\lambda\mu} \chi_\lambda,
\]
where \( \dim \lambda \) is the dimension of the representation \( \lambda \in \hat{G} \).

Since \( G \) is compact, every finite dimensional representation has a natural \( G \)-invariant Hermitian inner product so that the representation is unitary. In particular, we have
\[
|\chi_\lambda(x)| \leq |\chi_\lambda(1)| = \dim \lambda \quad \text{for every } x \in G.
\]
Therefore, if a sequence \( a_\lambda \in \mathbb{C} \) satisfies the convergence condition
\[
\sup_{x \in G} \left| \sum_{\lambda \in \hat{G}} a_\lambda \chi_\lambda(x) \right| \leq \sum_{\lambda \in \hat{G}} |a_\lambda| \dim \lambda < \infty,
\]
then the series $\sum_\lambda a_\lambda \chi_\lambda$ is uniformly and absolutely convergent and defines a $C^\infty$ class function on $G$. Note that every $L^2$ class function on $G$ can be expanded into a series of the form $\sum_\lambda a_\lambda \chi_\lambda$ with the convergence condition $\sum_\lambda |a_\lambda|^2 < \infty$. (If $G$ is a compact connected abelian group, i.e., a compact torus, then the expansion in terms of irreducible characters is the Fourier series expansion.) In the same way, a class distributions on $G$ is expanded into a series in $\chi_\lambda$ that is not necessarily absolutely convergent. For example, the $\delta$-function has an expansion
\begin{equation}
\delta(x) = \frac{1}{|G|} \sum_{\lambda \in \hat{G}} \dim \lambda \cdot \chi_\lambda(x).
\end{equation}
This infinite sum is convergent with respect to the strong topology of the topological linear space $\mathcal{D}'(G)$, and gives an alternative definition of the delta function. We note that the convolution relation $\delta \ast \delta = \delta$ is equivalent to Schur’s orthogonality relation (5.12).

The following family of class distributions parametrised by $x \in G$ are used in Section 3:
\begin{equation}
\eta_x(w) = \int_G \delta(yxy^{-1}w^{-1})dy.
\end{equation}
It is the distribution corresponding to the linear functional
$$g(w) \mapsto \int_G g(yxy^{-1})dy$$
for every smooth function $g(w)$. It is a class distribution because for every $u \in G$,
$$\int_G g(yxy^{-1})dy = \int_G g(uyxy^{-1}u^{-1})d(uy) = \int_G g(u(yxy^{-1})u^{-1})dy.$$
The character expansion of $\eta_x(w)$ as a distribution in $w$ has the following simple form:
\begin{equation}
\eta_x(w) = \sum_{\lambda \in \hat{G}} \overline{\chi_\lambda(x)} \chi_\lambda(w).
\end{equation}

Distributions cannot be evaluated in a meaningful way in general. For example, $\delta(1)$ is not defined. Similarly, restriction of a distribution defined on $G$ to a closed subset is generally meaningless. However, a distribution can always be localized to any open subset of $G$. Let $U \subset G$ be an open subset and $\mathcal{D}(U)$ the Fréchet space of $C^\infty$ functions on $U$ with compact support. The localization of a distribution on $G$ to $U$,
\begin{equation}
\mathcal{D}'(G) \ni f \mapsto f|_U \in \mathcal{D}'(U),
\end{equation}
is defined by
$$\int_U (f|_U)(x)g(x)dx = \int_G f(x)\tilde{g}(x)dx$$
for every $g(x) \in \mathcal{D}(U)$, where $\tilde{g}(x)$ is the extension of $g(x)$ as a $C^\infty$ function on $G$ satisfying
$$\tilde{g}(x) = \begin{cases} g(x) & x \in U \\ 0 & x \notin U \end{cases}.$$
Since $C^\infty(U) \subset \mathcal{D}'(U)$, we can compare $f|_U$ with a smooth function on $U$. A distribution $f \in \mathcal{D}'(G)$ is said to be regular at $w \in G$ if there is an open neighborhood $U \ni w$ such that the localization $f|_U$ of $f$ is a $C^\infty$ function on $U$. If $f$ is regular at $w \in G$, then the value $f(w) \in \mathbb{C}$ of $f$ at $w$ is well-defined.
6. Representation Varieties and Cell-Decompositions of a Surface

We have observed that for every closed surface \( S \) and a compact Lie groups \( G \), the quantity \(|G|^{\chi(S)}\text{Hom}(\pi_1(S), G)|^{-1}\) is a natural object that does not depend on the choice of an invariant measure on \( G \). In a recent work [35, 36], it has been noted that the same quantity for a finite group appears in the graphical expansion of a certain integral over the group algebra of the finite group. In this section we present a heuristic argument that relates the volume of the representation variety of a surface group to each term of the Feynman diagram expansion of a hypothetical “integral” over the group algebra of compact Lie groups. The argument is based on an alternative definition of the volume of the representation variety of a surface group in terms of a cell-decomposition of the surface.

The 1-skeleton of a cell-decomposition of a closed connected surface is a graph drawn on the surface. Such graphs are known as maps, fatgraphs, ribbon graphs, dessins d’enphants, or Möbius graphs, depending on the context of study. Fatgraphs and ribbon graphs are used effectively in the study of topological properties of moduli spaces of pointed Riemann surfaces and related topics (see for example, [4, 18, 19, 22, 39, 40, 41, 42, 53]). Dessins d’enphants appear in the context of algebraic curves defined over the field of algebraic numbers (cf. [3, 17, 32, 33, 45, 46]). Maps are drawn on a topological surface and are studied from the point of view of map coloring theorems (see for example, [16, 44]), where non-orientable surfaces are also considered. Graphs on non-orientable surfaces play an important role in the study of moduli spaces of real algebraic curves [15]. In this section we use the notion of Möbius graphs following [34, 36] to emphasize that our graphs are on unoriented surfaces, which may or may not be orientable.

A ribbon graph is a graph with a cyclic order given at each vertex to the set of half-deges incident to it. Equivalently, it is a graph drawn on a closed oriented surface [32]. Vertices with cyclic orders are connected by ribbon like edges to form a ribbon graph [22]. In order to deal with a graph on a non-orientable surface, it is necessary to allow ribbons with a twist. We can use the group \( \mathbb{S}_2 = \{\pm 1\} \) to indicate a twist of an edge with the value \(-1\) and no twist with \(1\). We are thus led to consider a ribbon graph with \( \mathbb{S}_2 \)-color at every edge. A vertex flip operation of an edge-colored ribbon graph is a move at a vertex \( V \) of a graph that reverses the cyclic order at \( V \) and interchanges the color of the edge that is incident to \( V \). If an edge makes a loop and incident to \( V \) twice, then its color is changed twice and hence preserved. Two \( \mathbb{S}_2 \)-colored ribbon graphs are equivalent if one is brought to the other by a sequence of vertex flip operations. A Möbius graph is the equivalence class of a \( \mathbb{S}_2 \)-colored ribbon graph [36].

Let \( \Gamma_c \) be a connected \( \mathbb{S}_2 \)-colored ribbon graph, and \( G \) a compact Lie group. For every vertex \( V \) of \( \Gamma_c \), we label the half-edges incident to \( V \) as \( (V_1, V_2, \ldots, V_n) \) according to the cyclic order, where \( n \) is the valence (or degree) of the vertex. We call the expression

\[
\Phi(V) = \delta(x_{V_1} \cdots x_{V_n}) \in \mathcal{D}'(G^n)
\]

the vertex contribution. It is invariant under the cyclic permutation of the half-edges. Suppose an edge \( E \) is incident to vertices \( V \) and \( W \) of \( \Gamma_c \), and consists of two half-edges \( V_i \) and \( W_j \). The propagator of the edge \( E \) is the expression

\[
\Phi(E) = \delta(x_{V_i}x_{W_j}^c) \in \mathcal{D}'(G^2),
\]

where \( c \in \mathbb{S}_2 = \{\pm 1\} \) is the color of \( E \). A few remarks are necessary for the definition of a vertex contribution and a propagator as distributions. For every \( n \geq 2 \), the multiplication

\[
m_n : G^n \ni (x_1, \ldots, x_n) \mapsto x_1 \cdots x_n \in G
\]
is a real analytic map and
\[ \psi_n : G^n \ni (x_1, \ldots, x_{n-1}, x_n) \mapsto ((x_1, \ldots, x_{n-1}), (x_1 \cdots x_{n-1} \cdot x_n) \in G^{n-1} \times G \]
is a real analytic isomorphism. The Jacobian \( J(x_1, \ldots, x_{n-1}, y) \) of the isomorphism \( \psi_n \) gives the push-forward measure
\[ (\psi_n)_* (dx_1 \cdots dx_n) = J(x_1, \ldots, x_{n-1}, y) dx_1 \cdots dx_{n-1} dy. \]
The vertex contribution (6.1) is a distribution defined by the continuous linear functional
\[ C^\infty(G^n) \ni f(x_1, \ldots, x_n) \mapsto \int_{G^{n-1}} J(x_1, \ldots, x_{n-1}, 1)f(x_1, \ldots, x_{n-1}, (x_1 \cdots x_{n-1})^{-1}) dx_1 \cdots dx_{n-1} \in \mathbb{C}. \]
The propagator as a distribution is defined similarly. We note that the product of all vertex contributions and the product of all propagators
\[ \prod_{V \in \Gamma_c} \Phi(V) \quad \text{and} \quad \prod_{E \in \Gamma_c} \Phi(E) \]
are well-defined distributions in \( \mathcal{D}'(G^{2e}) \), where \( e = e(\Gamma_c) \) is the number of edges of the graph.

Now we come to the heuristic part of the argument. Let us “define”
\[ G_{\Gamma_c} = \int_{G^{2e(\Gamma_c)}} \prod_{V \in \Gamma_c} \Phi(V) \prod_{E \in \Gamma_c} \Phi(E) dx, \]
where \( dx \) is the product measure on \( G^{2e(\Gamma_c)} \). Although these products are individually well-defined distributions on the space \( G^{2e(\Gamma_c)} \), their product does not make sense in general. For now, let us suppose that \( G_{\Gamma_c} \) is legally defined. Then the first thing we can show is that it is invariant under vertex flip operations. To see this, choose a vertex \( V \) at which we perform the vertex flip. Suppose it is incident to half-edges \( V_1, \ldots, V_n \) in this cyclic order. We denote by \( V' \) the other half-edge connected to \( V \). Of course we do not mean they are all incident to another vertex labeled by \( V' \). This is just a notation for the other half. Then the vertex flip invariance follows from
\[ \delta(x_{V_1}^{-c} \cdots x_{V_n}^{-c}) = \int_{G^n} \delta(x_{V_1} \cdots x_{V_n}) \delta(x_{V_1} x_{V_1'}^{-c} \cdots x_{V_n} x_{V_n'}^{-c}) dx_{V_1} \cdots dx_{V_n} \]
\[ = \int_{G^n} \delta(x_{V_1} \cdots x_{V_n}) \delta(x_{V_1}^{-c} x_{V_1'} \cdots x_{V_n}^{-c} x_{V_n'}) dx_{V_1} \cdots dx_{V_n} \]
\[ = \delta(x_{V_1'} \cdots x_{V_n'}), \]
where the first and the last delta functions are the same because of (5.8). Notice that the vertex flip invariance of \( G_{\Gamma_c} \) implies that this quantity is associated to a Möbius graph. So we define
\[ (6.3) \quad G_\Gamma = \int_{G^{2e(\Gamma)}} \prod_{V \in \Gamma} \Phi(V) \prod_{E \in \Gamma} \Phi(E) dx \]
for every Möbius graph \( \Gamma \).

The second property we can show, still assuming (6.3) being well-defined, is that it is invariant under an edge contraction. As explained in [32, 36], an edge contraction of a graph removes an edge that is incident to two distinct vertices and put these vertices together. If the edge is not twisted, then this is exactly the same edge contraction of [32]. If the edge is
twisted, then we first apply a vertex flip operation to one of the vertices to untwist the edge, and then proceed with the usual edge contraction. To see the invariance, suppose $E$ is the edge to be removed, which is incident to two vertices $V \neq W$. Without loss of generality, we assume that $E$ is not twisted, as mentioned above. Let us denote by $\delta(xy)$ the propagator. Using cyclic permutations of vertex contributions, we can assume that $\delta(vx)$ and $\delta(yw)$ are the vertex contributions at $V$ and $W$, respectively, where $v$ and $w$ are products of group elements. The invariance under the edge contraction then follows from

$$\int_{G^2} \delta(vx) \delta(xy) \delta(yw) dx dy = \int_G \delta(vy^{-1}) \delta(yw) dy = \delta(vw).$$

Note that $\delta(vw)$ is the vertex contribution of the new vertex obtained by joining $V$ and $W$ together.

Recall the fact that every connected Möbius graph $\Gamma$ determines a unique closed surface $S_\Gamma$ and its cell-decomposition. Let $v(\Gamma)$, $e(\Gamma)$, and $f(\Gamma)$ denote the number of 0, 1, and 2-cells, respectively. The Euler characteristic of the surface is given by

$$\chi(S_\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma).$$

If $\Gamma$ is orientable, then $S_\Gamma$ is an orientable surface of genus $g$ such that $\chi(S_\Gamma) = 2 - 2g$, and if $\Gamma$ is non-orientable, then $S_\Gamma$ is a non-orientable surface of cross-cap genus $k$ with $\chi(S_\Gamma) = 2 - k$. Notice that the edge contraction preserves the Euler characteristic and the number $f(\Gamma)$ of 2-cells. Using the same argument of [36], we can calculate $G_\Gamma$ for every graph. If the graph is orientable of genus $\geq 1$, then

$$G_\Gamma = |G|^{f(\Gamma) - 1} \int_{G^{2g}} \delta([x_1, y_1] \cdots [x_g, y_g]) dx_1 dy_1 \cdots dx_g dy_g,$$

and if it is non-orientable of cross-cap genus $k$, then

$$G_\Gamma = |G|^{f(\Gamma) - 1} \int_{G^k} \delta(x_1^2 \cdots x_k^2) dx_1 \cdots dx_k.$$

Regardless the orientation of the graph, we have a general formula

$$G_\Gamma = |G|^{f(\Gamma) - 1} |\text{Hom}(\pi_1(S_\Gamma), G)|$$

if the expressions (6.5) and (6.6) are well-defined. In the sense of (6.7), we have the alternative “definition” (6.3) of the volume of the representation variety.

It is clear from the invariance under edge contraction that $G_\Gamma$ of (6.3) is always ill-defined if $\chi(S_\Gamma) = 2$. As we have seen in Section 4, the integral (6.6) is ill-defined in general for graphs with $\chi(S_\Gamma) = 1$. For an orientable graph of genus $g \geq 2$, the integral (6.5) makes sense only when the group $G$ is semisimple (see Section 3).

Everything we discussed in this section makes sense for a finite group $G$, and we have the Feynman diagram expansion formula

$$\log \int_{\{x \in \mathbb{R}[G] \mid x^* = x\}} e^{-\frac{1}{2} \langle x^2 \rangle} e^{-\sum_j \frac{i}{2} \langle x^j \rangle} d\mu(x) = \sum_{\Gamma \text{ connected Möbius graph}} \frac{1}{|\text{Aut}_M(\Gamma)|} G_\Gamma \prod_j v_j(\Gamma)$$

established in [35, 36], where $\langle \rangle = \frac{1}{|G|} \chi_{\text{reg}}$ is the normalized trace of the finite-dimensional von Neumann algebra $\mathbb{R}[G]$, which is the real group algebra of $G$, and $v_j(\Gamma)$ is the number of $j$-valent vertices of $\Gamma$. We note that if we use the regular representation itself instead of the normalized trace in the integral, then $G_\Gamma$ of the graphical expansion is replaced by the familiar $|G|^{\chi(S_\Gamma)}|\text{Hom}(\pi_1(S_\Gamma), G)|$. We refer to the above mentioned articles for
more explanation of the expansion formula. The point we wish to make here is that the RHS of (6.8) tends to be well-defined for a compact semisimple Lie group $G$ when the Euler characteristic $\chi(S)$ is negative. Then how should we consider the integral of the LHS for a compact Lie group? Is the ill-defined nature of the infinite dimensional integral concentrated on the non-hyperbolic surfaces? Namely, if we remove the contributions of the surfaces of non-negative Euler characteristics from the integral over the infinite-dimensional space, would then the integral become well-defined? If the integral over the real group algebra $L^2(G)$ or even $D'_R(G)$ could make sense in some way, then Peter-Weyl theory gives a decomposition of the integral into an infinite sum of matrix integrals, and the volume formulas (3.7) and (4.7) should follow from the comparison of terms corresponding to orientable and non-orientable surfaces in the Feynman diagram expansion of the integral, as it was the case for a finite group [35, 36].

The alternative definition (6.3) suggests that this subject is related to the study of tensor categories and modular functors [2, 7].

References


