M17 MAT25-21 HOMEWORK 1 SOLUTIONS

1. To Hand In

Exercise 1: Distributive Properties of Finite Unions and Intersections

Let A, B, and C be sets. Show that the following properties hold.
(b) (*Intersection distributes over union*). $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We show containment in both directions.

- ($\subseteq$). If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. Hence either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$, and hence $x \in (A \cap B) \cup (A \cap C)$. On the other hand, if $x \in C$, then $x \in A \cap C$, and hence $x \in (A \cap B) \cup (A \cap C)$. So in either case, $x \in (A \cap B) \cup (A \cap C)$, as desired.
- ($\supseteq$). If $x \in (A \cap B) \cup (A \cap C)$, then either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, hence $x \in A$ and $x \in B \cup C$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$, hence $x \in A$ and $x \in B \cup C$, so $x \in A \cap (B \cup C)$. In either case we have that $x \in A \cap (B \cup C)$, as desired.

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Exercise 2: Distributive Properties of Infinite Unions and Intersections

Let A and B_n for each $n \in \mathbb{N}$ be sets. Show that the following properties hold.
(a) (*Union distributes over infinite intersection*). $A \cup (\bigcap_{n=1}^{\infty} B_n) = \bigcap_{n=1}^{\infty}(A \cup B_n)$.
(b) (*Intersection distributes over infinite union*). $A \cap (\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty}(A \cap B_n)$.

Proof.

(a)

- ($\subseteq$). If $x \in A \cup (\bigcap_{n=1}^{\infty} B_n)$, then either $x \in A$ or $x \in B_n$ for all n. If $x \in A$, then $x \in A \cup B_n$ for all n, hence $x \in \bigcap_{n=1}^{\infty}(A \cup B_n)$. If $x \in B_n$ for all n, then $x \in A \cup B_n$ for all n, hence $x \in \bigcap_{n=1}^{\infty}(A \cup B_n)$ as well.
- ($\supseteq$). If $x \in \bigcap_{n=1}^{\infty}(A \cup B_n)$, then $x \in A \cup B_n$ for all n. If $x \in A$, then $x \in A \cup (\bigcap_{n=1}^{\infty} B_n)$. If $x \notin A$, then since $x \in A \cup B_n$ for all n, it must be that $x \in B_n$ for all n. Therefore $x \in \bigcap_{n=1}^{\infty} B_n$, and therefore $x \in A \cup (\bigcap_{n=1}^{\infty} B_n)$.

(b)

- ($\subseteq$). If $x \in A \cap (\bigcup_{n=1}^{\infty} B_n)$, then $x \in A$ and $x \in B_n$ for some n. Therefore $x \in A \cap B_n$ for some n, hence $x \in \bigcup_{n=1}^{\infty}(A \cap B_n)$.
- ($\supseteq$). If $x \in \bigcup_{n=1}^{\infty}(A \cap B_n)$, then $x \in A \cap B_n$ for some n, and therefore $x \in A$ and $x \in B_n$ for some n. Hence $x \in A \cap (\bigcup_{n=1}^{\infty} B_n)$.

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Exercise 3: Example of Infinite Unions (Abbott Exercise 1.2.4)

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Proof. For $n \in \mathbb{N}$, define

$$A_n = \{2^{n-1}, (3)(2^{n-1}), (5)(2^{n-1}), (7)(2^{n-1}), \dots\}$$

I.e. A_n is all odd multiples of 2^{n-1}. We must show that these sets satisfy the desired properties.

- (*Infinite Number of Elemenets*). It is clear that the set $A_n = \{2^{n-1}, (3)(2^{n-1}), (5)(2^{n-1}), (7)(2^{n-1}), \dots\}$ has infinitely many elements.

- (*Disjoint*). Given A_n and A_m with $n \neq m$, we can assume, without loss of generality, that $n < m$. Suppose that there existed some $x \in A_n \cap A_m$. Then by definition of these sets, there exists some odd numbers k and ℓ such that $x = 2^{n-1}k = 2^{m-1}\ell$. However since $n < m$, we have that $n \leq m - 1$, and therefore we can write $2^{m-1} = (2^n)(2^i)$ with $i \geq 0$. Hence we have $2^{n-1}k = 2^n 2^i \ell$. Dividing both sides by 2^{n-1} yields $k = (2)(2^i)\ell$, which contradicts the assumption that k is odd. Therefore $A_n \cap A_m = \emptyset$.
- (*Union is* $\mathbb{N}$). We want to show that $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.
  - ($\subseteq$). Since each A_n is a subset of $\mathbb{N}$, the union of these sets is a subset of $\mathbb{N}$ as well.
  - ($\supseteq$). Given any $x \in \mathbb{N}$, we can write $x = 2^{n-1}k$ for some $n \in \mathbb{N}$ where k is odd. Then $x \in A_n$, as desired.

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Exercise 4: Finite De Morgan's Laws (Abbott Exercise 1.2.5)

Let A and B be subsets of a set X. Show that the following set equalities hold.

(a) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.
(b) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

These properties are sometimes called *De Morgan's Laws.*

Proof. (a)

- ($\subseteq$). If $x \in X \setminus (A \cap B)$ this means that x is not in $A \cap B$. Therefore either $x \notin A$ or $x \notin B$. Hence either $x \in X \setminus A$ or $x \in X \setminus B$. It follows that $(X \setminus A) \cup (X \setminus B)$.
- ($\supseteq$). If $x \in (X \setminus A) \cup (X \setminus B)$ then either $x \in X \setminus A$ or $x \in X \setminus B$. Hence either $x \notin A$ or $x \notin B$. Therefore $x \notin A \cap B$, so $x \in X \setminus (A \cap B)$.

(b)

- ($\subseteq$). If $x \in X \setminus (A \cup B)$, then $x \notin A \cup B$. Hence $x \notin A$ and $x \notin B$, so $x \in X \setminus A$ and $x \in X \setminus B$. Therefore $x \in (X \setminus A) \cap (X \setminus B)$.
- ($\supseteq$). If $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X \setminus A$ and $x \in X \setminus B$. Hence $x \notin A$ and $x \notin B$, so we have that $x \notin A \cup B$, and therefore $x \in X \setminus (A \cup B)$.

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Exercise 5: Infinite De Morgan's Laws

Let A_n for each $n \in \mathbb{N}$ be subsets of a set X. Show that the following set equalities hold.

(a) $X \setminus (\bigcap_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} (X \setminus A_n)$.
(b) $X \setminus (\bigcup_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} (X \setminus A_n)$.

These properties are also referred to as *De Morgan's Laws.*

Proof. (a)

- ($\subseteq$). If $x \in X \setminus (\bigcap_{n=1}^{\infty} A_n)$ then $x \notin \bigcap_{n=1}^{\infty} A_n$, therefore there exists some n such that $x \notin A_n$. Hence $x \in X \setminus A_n$ for some n, and therefore $x \in \bigcup_{n=1}^{\infty} (X \setminus A_n)$.
- ($\supseteq$). If $x \in \bigcup_{n=1}^{\infty} (X \setminus A_n)$, then $x \in X \setminus A_n$ for some n, and therefore $x \notin A_n$ for some n, so $x \notin \bigcap_{n=1}^{\infty} A_n$, and therefore $x \in X \setminus (\bigcap_{n=1}^{\infty} A_n)$.

(b)

- ($\subseteq$). If $x \in X \setminus (\bigcup_{n=1}^{\infty} A_n)$, then $x \notin \bigcup_{n=1}^{\infty} A_n$, so $x \notin A_n$ for all n. Hence $x \in X \setminus A_n$ for all n, and therefore $x \in \bigcap_{n=1}^{\infty} (X \setminus A_n)$.
- ($\supseteq$). If $x \in \bigcap_{n=1}^{\infty} (X \setminus A_n)$, then $x \in X \setminus A_n$ for all n, so $x \notin A_n$ for all n. Therefore $x \notin \bigcup_{n=1}^{\infty} A_n$, and therefore $x \in X \setminus (\bigcup_{n=1}^{\infty} A_n)$.

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2. Practice Problems (Don't hand these in)

Exercise 6: Examples of Set Containment

For each part below, give examples (different from the ones given in the lecture notes) of sets A and B that satisfy the required conditions.

(a) $A \subseteq B$.
(b) $A \subsetneq B$.
(c) $A \nsubseteq B$ and $B \nsubseteq A$.

Exercise 7: Uniqueness of the Empty Set

Show that the empty set is unique. That is, show that if A is any other set which contains no elements, then $\emptyset = A$.

Exercise 8: Further Properties of Unions and Intersections of Sets (Abbott Exercise 1.2.3)

Decide which of the following represent true statements about the nature of sets. If a statement is true, prove it. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are finite, nonempty sets, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
(c) $A \cap (B \cup C) = (A \cap B) \cup C$.
(d) $A \cap (B \cap C) = (A \cap B) \cap C$.
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 9: Cartesian Product of Unions and Intersections

For the following statements, prove or find a counterexample. Let X and Y be sets, with $A, B \subseteq X$ and $C, D \subseteq Y$.

(a) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.
(b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$.

Exercise 10: Determining if a Relation is an Equivalence Relation

Consider the relation S on $\mathbb{N} \times \mathbb{N}$ defined by $S := \{(x, y) : x, y \in \mathbb{N} \text{ and } x < y\} \subseteq \mathbb{N} \times \mathbb{N}$. Is this relation an **equivalence relation**?