1. To Hand In

Supremums and Infimums.

Exercise 1: Supremum Contained in the Set (Abbott Exercise 1.3.7)

Let $F$ be an ordered field, and let $A \subseteq F$. Show that if $a$ is an upper bound for $A$ and $a \in A$, then $a = \sup A$.

**Proof.** We need to show that $a$ satisfies (S1) and (S2).

(S1) By hypothesis, $a$ is an upper bound for $A$.

(S2) Suppose that $u \in F$ is some other upper bound. Then $x \leq u$ for all $x \in A$. Since $a \in A$, it follows, in particular, that $a \leq u$. Hence $a$ is the least upper bound of $A$.

\[ \blacksquare \]

The Least Upper Bound Property.

Exercise 2: Relationship between Supremums and Infimums (Abbott Exercise 1.3.3)

Let $F$ be an ordered field with the least upper bound property, and let $A \subseteq F$ be nonempty and bounded below, and define $B = \{ b \in F : b \text{ is a lower bound for } A \}$. Show that $\sup B = \inf A$.

**Proof.** First of all, note that $A$ is nonempty, and therefore there exists some $a \in A$, which satisfies $b \leq a$ for all $b \in B$, because each $b$ is a lower bound for $A$. Then $B$ is bounded above by $a$, hence because $F$ has the least upper bound property, $B$ has a supremum. Let $x = \sup B$. We just need to show that $x$ satisfies axioms (I1) and (I2) of the definition of the infimum.

(I1) Given any $a \in A$, $b \leq a$ for all $b \in B$. Therefore $a$ is an upper bound for $B$. By (S2), since $\sup B$ is the least upper bound of $B$, we have that $\sup B \leq a$. Since $a \in A$ was arbitrary, it follows that $\sup B \leq a$ for all $a \in A$, hence $\sup B$ is a lower bound for $A$.

(I2) Given any other lower bound $\ell$ for $A$, note that by definition of $B$, $\ell \in B$. Therefore $\ell \leq \sup B$. Hence $\sup B$ is the greatest lower bound of $A$.

Hence $x$ is the infimum of $A$, as desired.

\[ \blacksquare \]

Exercise 3: Algebraic Properties of Supremums (Abbott Exercise 1.3.5 and 1.3.6)

Let $F$ be an ordered field with the least upper bound property and let $A, B \subseteq F$ be nonempty subsets which are bounded above. Let $c \in F$, and define $c + A := \{ c + a : a \in A \}$, $cA := \{ ca : a \in A \}$, and $A + B = \{ a + b : a \in A, b \in B \}$.

(b) Show that if $c \geq 0$ then $\sup(cA) = c \sup(A)$.

(c) Show that $\sup(A + B) = \sup(A) + \sup(B)$.

**Proof.**

(b) We’ll show that $\sup(cA) \leq c \sup(A)$ and that $c \sup(A) \leq \sup(cA)$.
• \((\sup(cA) \leq c\sup(A))\). Given any \(ca \in cA\), since \(a \in A\), we have that \(a \leq \sup(A)\). Since \(c \geq 0\), this implies that \(ca \leq c\sup(A)\). Since \(ca \in cA\) was arbitrary, it follows that \(c\sup(A)\) is an upper bound for \(cA\). Hence by (S2) applied to \(c\sup(A)\), we have that \(c\sup(A) \leq c\sup(A)\).

• \((c\sup(A) \leq \sup(cA))\). We consider two cases.
  
  – \((c = 0)\). If \(c = 0\), the \(cA = \{0\}\) and \(c\sup(A) = 0\). In this case it is clear that \(c\sup(A) \leq \sup(cA)\).
  
  – \((c > 0)\). By (S1) applied to \(cA\), we have that for any \(ca \in cA\), \(ca \leq c\sup(A)\). Dividing both sides by \(c > 0\), we obtain that \(a \leq \frac{1}{c}\sup(cA)\). Since this is true for all \(a \in A\), we have that \(\frac{1}{c}\sup(cA)\) is an upper bound for \(A\). Therefore by (S2) applied to \(A\), it follows that \(\sup(A) \leq \frac{1}{c}\sup(cA)\).

Multiplying both sides by \(c > 0\), we obtain that \(c\sup(A) \leq \sup(cA)\), as desired.

(c) We’ll show that \(\sup(A + B) \leq \sup(A) + \sup(B)\) and that \(\sup(A) + \sup(B) \leq \sup(A + B)\).

• \((\sup(A + B) \leq \sup(A) + \sup(B))\). Given any \(a + b \in A + B\), we have that \(a \leq \sup(A)\) and \(b \leq \sup(B)\), so adding these two inequalities together yields \(a + b \leq \sup(A) + \sup(B)\). Since this is true for all \(a + b \in A + B\), (S2) applied to \(A + B\) yields \(\sup(A + B) \leq \sup(A) + \sup(B)\).

• \((\sup(A) + \sup(B) \leq \sup(A + B))\). Given any \(a \in A\) and \(b \in B\), we have \(a + b \leq \sup(A + B)\), hence \(a \leq \sup(A + B) - b\). For any \(b \in B\), this is true for all \(a \in A\), so we have that \(\sup(A) \leq \sup(A + B) - b\). But this is now true for any \(b \in B\), hence we have \(b \leq \sup(A + B) - \sup(A)\) for all \(b \in B\). Hence \(\sup(B) \leq \sup(A + B) - \sup(A)\), which implies that \(\sup(A) + \sup(B) \leq \sup(A + B)\), as desired.

The Real Numbers: Existence and Preliminary Results.

**Exercise 4: The Absolute Value Function (Abbott Example 1.2.5/Exercise 1.2.6)**

Define a function, called the absolute value function, as follows.

\[
| \cdot | : \mathbb{R} \to \mathbb{R} \\
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

Show that the absolute value function satisfies the following properties for all \(a, b \in \mathbb{R}\).

(a) \(|ab| = |a||b|\).

(b) (Triangle Inequality). \(|a + b| \leq |a| + |b|\).

(c) (Nonnegativity). For all \(a \in \mathbb{R}\), \(|a| \geq 0\).

(d) \(|a| < b\) if and only if \(-b < a < b\).

(e) (Reverse Triangle Inequality). \(||a| - |b|| \leq |a - b|\).

**Proof.**

(a) Without loss of generality, we consider three cases.

• \((a \geq 0, b \geq 0)\). Then \(ab \geq 0\), so \(|ab| = ab\). Also \(|a| = a\) and \(|b| = b\), hence \(|ab| = ab\).

• \((a \geq 0, b \leq 0)\). Then \(ab \leq 0\), so \(|ab| = -ab\). Also \(|a| = a\) and \(|b| = -b\), hence \(|ab| = -ab\).

• \((a \leq 0, b \leq 0)\). Then \(ab \geq 0\), so \(|ab| = ab\). Also \(|a| = -a\) and \(|b| = -b\), hence \(|ab| = (-a)(-b) = ab\).

(b) Note that for any \(c \in \mathbb{R}\), \(c \leq |c|\). Then compute

\[
|a + b|^2 = (a + b)^2 = a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2.
\]

Additionally, given any \(c, d \in \mathbb{R}\) with \(c, d \geq 0\), if \(c^2 \leq d^2\) then \(c \leq d\), since if \(d < c\), it would follow that \(d^2 < c^2\) contradicting that \(c^2 \leq d^2\). Hence we have \(|a + b|, |a| + |b| \geq 0\), and \(|a + b|^2 \leq (|a| + |b|)^2\), which implies that \(|a + b| \leq |a| + |b|\), as desired.

(c) If \(a \geq 0\), the \(|a| = a \geq 0\). If \(a < 0\), then \(|a| = -a > 0\).

(d) \((\Rightarrow)\). Suppose that \(|a| < b\).

  1. \((a \geq 0)\). If \(a \geq 0\), then \(-b < 0 \leq a = |a| < b\), as desired.

  2. \((a < 0)\). If \(a < 0\), then \(-a > 0\) and \(|a| = -a\). Hence \(a < -a = |a| < b\). This implies that \(a < b\), and that \(-a < b\), which shows that \(-b < a\), so we have \(-b < a < b\), as desired.

• \((\Leftarrow)\). Suppose that \(-b < a < b\).
– \(a \geq 0\). If \(a \geq 0\), then \(|a| = a < b\), as desired.

– \((a < 0)\). If \(a < 0\), then since \(-b < a\) implies \(-a < b\), we have \(|a| = -a < b\) as desired.

(e) Compute

\[
|a| = |a - b + b| \leq |a - b| + |b| \\
\implies |a| - |b| \leq |a - b|,
\]

and

\[
|b| = |b - a + a| \leq |a - b| + |a| \\
\implies |b| - |a| \leq |a - b| \\
\implies -|a - b| \leq |a| - |b|.
\]

Putting these together, and using part (d), we obtain that

\[
-|a - b| \leq |a| - |b| \leq |a - b| \\
\implies ||a| - |b|| \leq |a - b|.
\]

\[\blacksquare\]

Consequences of Completeness of \(\mathbb{R}\).

**Exercise 5: Intersection of Open Intervals (Abbott Exercise 1.4.3)**

Consider, for each \(n \in \mathbb{N}\), the open intervals \((0, 1/n) \subseteq \mathbb{R}\). Prove that \(\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset\). Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

**Proof.** Suppose, for contradiction, that \(x \in \bigcap_{n=1}^{\infty} (0, 1/n)\). Then \(x \in (0, 1/n)\) for all \(n \in \mathbb{N}\), and hence \(0 < x < 1/n\) for all \(n \in \mathbb{N}\). However by the Archimedean property of \(\mathbb{R}\), there exists some \(m \in \mathbb{N}\) satisfying \(0 < 1/m < x\). Hence we obtain a contradiction, which implies that \(\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset\). \[\blacksquare\]

Existence of Square Roots in \(\mathbb{R}\).

**Exercise 6: Existence of Square Roots in \(\mathbb{R}\)**

(a) Modify the proof of the existence of \(\sqrt{2}\) in \(\mathbb{R}\) slightly to show that for any \(a \in \mathbb{R}\) with \(a \geq 0\), there exists \(b \in \mathbb{R}\) such that \(b^2 = a\). This is often stated as the existence of square roots in \(\mathbb{R}\).

(b) Show that if \(b^2 = a\), then \((-b)^2 = a\).

(c) Show that for \(a, b, c \in \mathbb{R}\), if \(b^2 = a\) and \(c^2 = a\), then either \(b = c\) or \(b = -c\).

(d) Show that for any \(a \in \mathbb{R}\) with \(a \geq 0\), there always exists a unique non-negative real number \(b\) such that \(b^2 = a\).

(e) Define a function \(\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}\) by letting \(\sqrt{a}\) be the unique non-negative real number \(b\) with \(b^2 = a\), whose existence is guaranteed by part (d).

(f) Prove that \(\sqrt{\cdot}\) is strictly increasing, meaning if \(0 \leq a < b\), then \(\sqrt{a} < \sqrt{b}\).

**Proof.** (a) In this case, we’ll modify the proof of the existence of \(\sqrt{2}\) slightly by considering four cases.
The Limit of a Sequence.

Exercise 7: Equivalence of Definitions of Convergence of a Sequence

Given a sequence \((a_n)\) of real numbers and a point \(a \in \mathbb{R}\), show that the following are equivalent.

(a) (Convergence of a Sequence). For every \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\), such that whenever \(n \geq N\), it follows that \(|a_n - a| < \epsilon\).

(b) (Convergence of a Sequence: Topological Version). Given any \(\epsilon\)-neighborhood \(V_\epsilon(a)\) of \(a\), there exists a point in the sequence after which all of the terms are in \(V_\epsilon(a)\).

(c) Every \(\epsilon\)-neighborhood of \(a\) contains all but a finite number of the terms of \((a_n)\).
Proof. \( \bullet ((a) \implies (b)) \). Let \( V_\epsilon(a) \) be an arbitrary \( \epsilon \)-neighborhood of \( a \). Then by (a), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( |a_n - a| < \epsilon \). Then \( a_N \) is a point in the sequence after which \( |a_n - a| < \epsilon \), which implies that \( a_n \in V_\epsilon(a) \).

\( \bullet ((b) \implies (c)) \). Let \( V_\epsilon(a) \) be an arbitrary \( \epsilon \)-neighborhood of \( a \). By part (b), there exists a point, which we’ll label as \( a_N \), in the sequence after which all of the terms of \( (a_n) \) are in \( V_\epsilon(a) \). Therefore for \( n \geq N \), \( a_n \in V_\epsilon(a) \), hence the set of points in the sequence which are not in \( V_\epsilon(a) \) is a subset of \( \{a_1, a_2, \ldots, a_{N-1}\} \), and is therefore a finite set. Therefore all but a finite number of terms of the sequence are in \( V_\epsilon(a) \).

\( \bullet ((c) \implies (a)) \). Let \( \epsilon > 0 \) be arbitrary. Then by (c), all but a finite number of terms of the sequence are in \( V_\epsilon(a) \), so there is some point \( a_N \) in the sequence such that if \( n \geq N \), then \( a_n \in V_\epsilon(a) \), which implies that \( |a_n - a| < \epsilon \) for all \( n \geq N \).

\[ \blacksquare \]

**Exercise 8: Uniqueness of Limits (Abbott Theorem 2.2.7/Exercise 2.2.6)**

Prove that the limit of a sequence, when it exists, is unique. I.e. if \( (a_n) \) converges to \( a \) and also converges to \( b \), then \( a = b \).

Proof. Suppose, for contradiction, that \( (a_n) \to a \) and \( (a_n) \to b \), but \( a \neq b \). Without loss of generality, we can assume that \( a < b \). Then let \( \epsilon = \frac{b - a}{2} > 0 \). Since \( (a_n) \to a \), there exists \( N_1 \in \mathbb{N} \) such that if \( n \geq N_1 \), it follows that \( |a_n - a| < \epsilon \). Similarly, since \( (a_n) \to b \), there exists \( N_2 \in \mathbb{N} \) such that if \( n \geq N_2 \), then \( |a_n - b| < \epsilon \). Let \( N = \max(N_1, N_2) \). If \( n \geq N \), then \( n \geq N_1, N_2 \), hence we compute

\[ b - a \leq |a - b| \leq |a - a_n + a_n - b| \leq |a_n - a| + |a_n - b| < \frac{b - a}{2}, \]

which is a contradiction, since \( b > a \). Therefore \( a = b \).

\[ \blacksquare \]

**Exercise 9: Examples of Limits of Sequences (Abbott Exercise 2.2.2)**

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

\( (a) \) \( \lim_{n \to \infty} \frac{2n+1}{5n+4} = \frac{2}{5} \).

\( (b) \) \( \lim_{n \to \infty} \frac{2n^2}{n^2+3} = 0 \).

Proof. (a) Given \( \epsilon > 0 \) arbitrary, by the Archimedean property of \( \mathbb{R} \), we may choose \( N \in \mathbb{N} \) such that \( N > \frac{3 - 20 \epsilon}{25 \epsilon} \). Then if \( n \geq N \), we have \( n > \frac{3 - 20 \epsilon}{25 \epsilon} \), which implies that \( \frac{3}{25n+20} < \epsilon \). Hence for \( n \geq N \), we have

\[
\left| \frac{2n + 1}{5n + 4} - \frac{2}{5} \right| = \left| \frac{5(2n + 1) - 2(5n + 4)}{5(5n + 4)} \right| = \left| \frac{-3}{25n + 20} \right| = \frac{3}{25n + 20} < \epsilon.
\]

(b) Given \( \epsilon > 0 \) arbitrary, by the Archimedean property of \( \mathbb{R} \), we may choose \( N \in \mathbb{N} \) such that \( N > \frac{\epsilon}{2} \). Then if \( n \geq N \), we have that \( n > \frac{\epsilon}{2} \) and hence that \( \frac{2}{n} < \epsilon \). Then compute

\[
\left| \frac{2n^2}{n^2+3} \right| = \frac{2n^2}{n^2+3} \leq \frac{2n^2}{n^2} = \frac{2}{n} < \epsilon.
\]

\[ \blacksquare \]
Exercise 10: Abbott Exercise 2.2.4

Give an example of each of state that the request is impossible. For any that are impossible, prove it.
(a) A sequence with an infinite number of 1’s that does not converge to 1.
(b) A sequence with an infinite number of 1’s that converges to a limit not equal to 1.
(c) A divergent sequence such that for every \( n \in \mathbb{N} \), it is possible to find \( n \) consecutive 1’s somewhere in the sequence.

Proof.  
(a) The sequence \((1, 2, 1, 3, 1, 4, 1, 5, \ldots)\) has infinitely many 1’s, but is not bounded, and therefore cannot converge to 1.
(b) This is not possible. Suppose that \((a_n)\) had infinitely many 1’s, but converged to \( a \neq 1 \). Assume that \( 1 < a \) (the other case is handled similarly). Then let \( \epsilon = a - 1 > 0 \). Since \((a_n)\) converges to \( a \), there exists some \( N \in \mathbb{N} \) such that if \( n \geq N \), then \(|a_n - a| < a - 1\). This implies, in particular, that \(-a + 1 < a_n - a\), which implies that \( 1 < a_n \) for all \( n \geq N \). But this means that all but a finite number of the \( a_n \)'s are strictly greater than 1, a contradiction.
(c) The sequence \((1, 2, 1, 1, 3, 1, 1, 4, 1, 1, 1, 5, \ldots)\) has \( n \) consecutive 1's for any \( n \), but is unbounded, and therefore divergent.

\[ \blacksquare \]

2. Practice Problems (Don’t hand these in)

The Least Upper Bound Property.

Exercise 11: Supremums of Unions (Abbott Exercise 1.3.4)

Let \( F \) be an ordered field with the least upper bound property and let \( A_1, A_2, \ldots A_k, \ldots \) be a collection of nonempty subsets of \( F \), each of which is bounded above.
(a) Find a formula for \( \sup(A_1 \cup A_2) \).
(b) Extend this to a formula for \( \sup(\bigcup_{k=1}^{n} A_k) \).
(c) Further extend this to a formula for \( \sup(\bigcup_{k=1}^{\infty} A_k) \).

The Real Numbers: Existence and Preliminary Results.

Exercise 12: “\( \forall \epsilon \)” Proofs (Abbott Exercise 1.2.10)

Decide which of the following are true statements. If they are true, prove them, and if they are false, find a specific counterexample.
(a) Two real numbers satisfy \( a < b \) if and only if \( a < b + \epsilon \) for all \( \epsilon > 0 \).
(b) Two real numbers satisfy \( a < b \) if \( a < b + \epsilon \) for all \( \epsilon > 0 \).
(c) Two real numbers satisfy \( a \leq b \) if and only if \( a < b + \epsilon \) for all \( \epsilon > 0 \).

Exercise 13: Finding Supremums

Let \( a < b \) be real numbers. Find the supremums of the following sets, and prove your results.
(c) \([a, b) = \{ r \in \mathbb{R} \mid a \leq r < b \} \).
(d) \((a, b] = \{ r \in \mathbb{R} \mid a < r \leq b \} \).
The Limit of a Sequence.

**Exercise 14: Convergence of a Constant Sequence**

Let \( x \in \mathbb{R} \) be a real number, and define a sequence \((a_n)\) with \(a_n = x\) for all \(n \in \mathbb{N}\). Show that \((a_n)\) converges to \(x\).

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Divergence.

**Exercise 15: Divergence**

Show that the sequence \((n)_{n=1}^\infty = (1, 2, 3, 4, \ldots)\) diverges.

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Bounded Sequences.

**Exercise 16: Bounded Sequence**

Recall the definition of a bounded subset of an ordered field \(??\). Show that a sequence \((x_n)\) of real numbers is bounded if and only if the set of terms of \((x_n)\), i.e. \(\{x_n : n \in \mathbb{N}\}\), is a bounded subset of \(\mathbb{R}\).

**Exercise 17: Bounded Sequences**

Is it true that every bounded sequence is convergent? Prove or find a counterexample.