Double Series.

The exercises in this section will guide you to complete the proof of the following theorem:

**Theorem 1: Absolute Convergence of Iterated and Double Series (Abbott Theorem 2.8.1)**

Let \((a_{ij})\) be a double sequence. If the iterated series
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|
\]
converges, then both iterated series \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}\) and \(\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}\) converge to the same value \(A\), and the unordered series \(\sum_{i,j=1}^{\infty} a_{ij}\) converges as a square sum to \(A\) as well.

Let \((a_{ij})\) be a double sequence and suppose that the iterated series
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|
\]
converges. Define \(t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|\).

**Exercise 1: (Abbott Exercise 2.8.3)**

(a) Prove that the sequence \((t_{nn})\) converges.
(b) Use the fact that \((t_{nn})\) is a Cauchy sequence to argue that \((s_{nn})\) converges.

Now we show that the iterated sum \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}\) converges to \(S\).

**Solution 1.**

(a) By hypothesis, the iterated series \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|\) converges. This means that for all \(i \in \mathbb{N}\), the series \(\sum_{j=1}^{\infty} |a_{ij}|\) converges to some \(b_i\), and the series \(\sum_{i=1}^{\infty} b_i\) converges as well, say to \(C\).

Define \(t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|\). We claim that the sequence \((t_{mn})_{n=1}^{\infty}\) converges. Given any \(i \in \mathbb{N}\), note that the sequence of partial sums of \(\sum_{j=1}^{\infty} |a_{ij}|\) is increasing. Since \(\sum_{j=1}^{\infty} |a_{ij}| = b_i\), it follows that \(|a_{i1}| + \ldots + |a_{in}| \leq b_i\) for all \(n \in \mathbb{N}\). Furthermore, each \(b_i\) is nonnegative. So since \(\sum_{i=1}^{\infty} b_i = C\), we have that \(b_1 + \ldots + b_n \leq C\) for all \(n \in \mathbb{N}\). Therefore, we can compute

\[
t_{nn} = (|a_{11}| + \ldots + |a_{1n}|) + (|a_{21}| + \ldots + |a_{2n}|) + \ldots + (|a_{n1}| + \ldots + |a_{nn}|) \leq b_1 + b_2 + \ldots + b_n \leq C.
\]

Hence the sequence \((t_{nn})\) is bounded above by \(C\). Additionally, \((t_{nn})\) is increasing. Therefore by the monotone convergence theorem, \((t_{nn})\) converges.

(b) Now we claim that \((s_{nn})\) is Cauchy. Let \(\epsilon > 0\) be arbitrary. Since \((t_{nn})\) converges, it is Cauchy. Therefore there exists some \(N \in \mathbb{N}\) such that if \(n > m \geq N\), we have \(|t_{nn} - t_{mm}| < \epsilon\). Compute
Here (1) follows from the triangle inequality. Therefore if \( n > m \geq N \), we have that \( |s_{nn} - s_{mm}| < \epsilon \). So \((s_{nn})\) is a Cauchy sequence, and therefore converges. Let \( S = \lim_{n \to \infty} s_{nn} \).

**Exercise 2:** (Abbott Exercise 2.8.4)

(a) Let \( \epsilon > 0 \) be arbitrary and prove that there exists \( N_1 \in \mathbb{N} \) such that for all \( n, m \geq N_1 \), we have \( B - \epsilon/4 < t_{mn} \leq B \).

(b) Show that there exists \( N \in \mathbb{N} \) such that for all \( n, m \geq N \), we have

\[ |s_{mn} - S| < \epsilon/2. \]

**Solution 2.**

(a) Since \((t_{mn})\) is bounded by \( C \), for any \( m, n \in \mathbb{N} \), we let \( k = \max(m, n) \) and then we have that \( t_{mn} \leq t_{kk} \leq B \). Hence the set \( \{t_{mn} \mid m, n \in \mathbb{N}\} \) is bounded, and therefore we can let

\[ B = \sup \{t_{mn} \mid m, n \in \mathbb{N}\}. \]

Given any \( \epsilon > 0 \), there exists \( t_{\tilde{m}\tilde{n}} \) such that \( B - \epsilon/4 < t_{\tilde{m}\tilde{n}} \leq B \). Let \( N_1 = \max(\tilde{m}, \tilde{n}) \). Then we have that \( t_{\tilde{m}\tilde{n}} \leq t_{N_1N_1} \leq B \), and if \( n, m \geq N_1 \), then we have

\[ B - \epsilon/4 < t_{\tilde{m}\tilde{n}} \leq t_{N_1N_1} \leq t_{mn} \leq B. \]

Therefore we have found \( N_1 \in \mathbb{N} \) such that if \( n, m \geq N_1 \), then \( B - \epsilon/4 < t_{mn} \leq B \). As a consequence of this, we have that whenever \( n, m \geq N_1 \), it follows that \( |t_{mn} - t_{mm}| < \epsilon/4 \).

(b) Furthermore, since \((s_{nn}) \to S\), there exists \( N_2 \) such that if \( n \geq N \) then \( |s_{nn} - S| < \epsilon/4 \). Let \( N = \max(N_1, N_2) \). Then if \( n, m \geq N \), we may assume, without loss of generality, that \( n > m \). Then we compute

\[
|s_{mn} - S| = |s_{mn} - s_{mm} + s_{mm} - S| \\
\leq |s_{mn} - s_{mm}| + |s_{mm} - S| \\
(1) \leq |t_{mn} - t_{mm}| + |s_{mm} - S| \\
< \epsilon/4 + \epsilon/4 \\
= \epsilon/2
\]

Here, (1) follows from the triangle inequality.

By hypothesis, for each \( i \), the series \( \sum_{j=1}^{\infty} |a_{ij}| \) converges, and therefore the series \( \sum_{j=1}^{\infty} a_{ij} \) converges to some \( r_i \in \mathbb{R} \).
Exercise 3: (Abbott Exercise 2.8.5)

(a) Let $\epsilon > 0$ and $N$ be as in the previous exercise. Show that for all $m \geq N$ we have

$$|(r_1 + \ldots + r_m) - S| \leq \epsilon/2 < \epsilon.$$  

Conclude that the iterated sum $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij}$ converges to $S$.

(b) Show that the other iterated sum $\sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}$ also converges to $S$. Notice that the same argument can be used once it is established that, for each column $j$, the sum $\sum_{i=1}^\infty a_{ij}$ converges to some real number $c_j$.

Solution 3.

(a) Recall that by hypothesis, for each $i \in \mathbb{N}$, $\sum_{j=1}^\infty |a_{ij}|$ converges, so the series $\sum_{j=1}^\infty a_{ij}$ is absolutely convergent, and therefore convergent, say to $r_i$. Then for any $m \in \mathbb{N}$, the Algebraic Limit Theorem says that

$$\sum_{j=1}^\infty \sum_{i=1}^m a_{ij} = r_1 + \ldots + r_m.$$  

Since the sequence of partial sums of $\sum_{j=1}^\infty \sum_{i=1}^m a_{ij}$ is $(s_{mn})$, it follows that

$$\lim_{n \to \infty} s_{mn} = r_1 + \ldots + r_m.$$  

Therefore if $m \geq N$, then for all $n \geq N$, we have

$$-\epsilon/2 + S < s_{mn} < \epsilon/2 + S.$$  

Hence by the Order Limit Theorem

$$-\epsilon/2 + S \leq r_1 + \ldots + r_m \leq \epsilon/2 + S \implies |(r_1 + \ldots + r_m) - S| \leq \epsilon/2$$  

$$\implies |(r_1 + \ldots + r_m) - S| < \epsilon.$$  

Therefore it follows that $\sum_{i=1}^\infty r_i = S$. In other words, we have $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = S$, as desired.

(b) To show that the other iterated series converges to $S$, we first show that the $j$th column of the series converges for all $j \in \mathbb{N}$. Recall that summing along the rows gives $\sum_{j=1}^\infty |a_{ij}| = b_i$. This implies that $|a_{ij}| \leq b_i$. Since $\sum_{j=1}^\infty b_j$ converges, the Comparison Test then implies that $\sum_{j=1}^\infty |a_{ij}|$ converges as well, say to $c_j$. Then a similar argument to that above shows that $\sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij} = S$, as desired.

Products of Series.

Exercise 4: (Abbott Exercise 2.8.7)

Assume that $\sum_{i=1}^\infty a_i$ and $\sum_{j=1}^\infty b_j$ converge absolutely to $A$ and $B$ respectively.

(a) Prove that the iterated series $\sum_{i=1}^\infty \sum_{j=1}^\infty |a_i b_j|$ converges.

(b) Prove that the unordered series $\sum_{i,j=1}^{\infty} a_i b_j$ converges as a square sum to $AB$, and use it to show that the iterated series $\sum_{i=1}^\infty \sum_{j=1}^\infty a_i b_j$ and $\sum_{j=1}^\infty \sum_{i=1}^\infty a_i b_j$, as well as $\sum_{k=2}^\infty d_k$, all converge to $AB$. Here, as in the lecture notes, we define

$$d_k = \sum_{1 \leq i,j \leq k \atop i+j=k} a_i b_j$$  

$$= a_1 b_{k-1} + a_2 b_{k-2} + \ldots + a_{k-1} b_1.$$
Solution 4.
(a) Since $\sum_{j=1}^{\infty} b_j$ converges absolutely, it follows that $\sum_{j=1}^{\infty} |b_j|$ converges to some $D \in \mathbb{R}$, hence by the Algebraic Limit Theorem, for each $i \in \mathbb{N}$, $\sum_{j=1}^{\infty} |a_i| |b_j|$ converges to $|a_i|D$. But then because $\sum_{i=1}^{\infty} |a_i|$ converges to some $C \in \mathbb{R}$, the sum $\sum_{i=1}^{\infty} |a_i|D$ converges to $CD$. Therefore the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i|b_j$ converges to $CD$.

(b) The sequence $(s_{nn})$ is given by

$$s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j = (a_1 + \ldots + a_n)(b_1 + \ldots + b_n).$$

Since $\lim_{n \to \infty} (a_1 + \ldots + a_n) = A$ and $\lim_{n \to \infty} (b_1 + \ldots + b_n) = B$, it follows from the Algebraic Limit Theorem for sequences that $\lim_{n \to \infty} s_{nn} = AB$. Hence the unordered series $\sum_{i,j=1}^{\infty} a_i b_j$ converges as a square sum to $AB$. By the previous three Exercises, it then follows that the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j$ also converge to $AB$.

The sequence $(t_m)$ of partial sums of the series $\sum_{k=2}^{\infty} d_k$ satisfies

$$t_m = d_2 + \ldots + d_{m+1}$$

$$\implies s_{\ell,\ell} \leq t_m \leq s_{m+1,m+1}.$$  

for some $\ell \in \mathbb{N}$. Therefore by the squeeze theorem, it follows that $\lim_{m \to \infty} t_m = AB$.

Open sets, Closed Sets, and Limit Points.

Exercise 5: (Abbott Exercise 3.2.6)

Decide whether the following statements are true or false. If they’re true, prove them. If they are false, provide counter examples.

(a) An open set that contains every rational number must necessarily contain all of $\mathbb{R}$.

(b) The Nested Interval Property remains true if the term “closed interval” is replaced by “closed set”.

(c) Every nonempty open set contains a rational number.

(d) Every bounded, infinite, closed set contains a rational number.

Solution 5.  
(a) This is false. The set $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ is the union of two open intervals, and is therefore open, and it contains every real number except for $\sqrt{2}$, so it contains every rational number, however it does not contain $\sqrt{2}$, so it does not contain every real number.

(b) This is false. Let $F_1 = \mathbb{N}$, $F_2 = \mathbb{N} \setminus \{1\}$, $F_3 = \mathbb{N} \setminus \{1, 2\}$, and for any $n \in \mathbb{N}$, $F_n = \mathbb{N} \setminus \{1, 2, \ldots, n - 1\}$. A similar proof to that which we used to show that $\mathbb{N}$ is closed shows that $F_n$ is closed for all $n \in \mathbb{N}$. Furthermore $n \in F_n$, so $F_n$ is nonempty. Finally, we have that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$$

However for each $n \in \mathbb{N}$, we have that $n \notin F_{n+1}$, and therefore $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

(c) This is true. Let $O$ be a nonempty open set. Since it is nonempty, there exists $x \in O$, and since $O$ is open, it follows that there is some $\epsilon > 0$ such that $V_{\epsilon}(x) \subseteq O$. Then $x - \epsilon < x + \epsilon$, so by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some $r \in \mathbb{Q}$ with $x - \epsilon < r < x + \epsilon$, and therefore $r \in V_{\epsilon}(x) \subseteq O$.

(d) This is not true. Consider the set $\{\sqrt{2} + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{\sqrt{2}\}$. Then Problem 3 on the practice final tells us that this set is closed, and yet because $\sqrt{2}$ is irrational, so is $\sqrt{2} + \frac{1}{n}$ for all $n \in \mathbb{N}$. This follows from noting that if $\sqrt{2} + \frac{1}{n}$ were rational, then because $-\frac{1}{n}$ is also rational, and the sum of two rational numbers is rational, it follows that $\sqrt{2} \neq \sqrt{2} + \frac{1}{n} = \frac{1}{n} \in \mathbb{Q}$, a contradiction.
**Exercise 6: (Abbot Exercise 3.2.4)**

Let $A$ be nonempty and bounded above so that $\sup(A)$ exists.

(a) Show that $\sup(A) \in \overline{A}$.

(b) Can an open set contain its supremum?

**Solution 6.**

(a) If $\sup(A) \in A$, then $\sup(A) \in \overline{A}$ since $\overline{A} = A \cup L$ where $L$ is the set of limit points of $A$. If $\sup(A) \notin A$, then Problem 4 part (c) on the practice midterm shows that $\sup(A)$ is a limit point of $A$, and therefore $\sup(A) \in L$, so $\sup(A) \in \overline{A}$.

(b) No. Let $U$ be an open, nonempty, bounded above subset of $\mathbb{R}$. Then $s = \sup U$ exists. If $s \in U$, then because $U$ is open, there exists some $\epsilon > 0$ such that $V_\epsilon(s) \subseteq U$. However $s < s + \epsilon/2$, yet $s + \epsilon/2$ is contained in $V_\epsilon(s)$, and therefore in $U$, contradicting that $s$ is an upper bound for $U$. Therefore $s \notin U$, as desired.

**Exercise 7: (Abbot Exercise 3.2.8)**

Assume $A$ is an open set and $B$ is a closed set. Determine if the following sets are open, closed, both, or neither and prove your result.

(a) $A \cup \overline{B}$.

(b) $A \setminus B = \{x \in A \mid x \notin B\}$.

(c) $(A^c \cup B)^c$.

(d) $(A \cap B) \cup (A^c \cap B)$.

(e) $(\overline{A})^c \cap \overline{A}$.

**Solution 7.**

(a) (Closed). Always. For any set $C$, $\overline{C}$ is closed. Therefore $\overline{A \cup B}$ is always closed.

(b) (Open). Sometimes. If we let $A = B = \mathbb{R}$, then we have $\overline{A \cap B} = \mathbb{R}$ which is open. On the other hand, if we let $A = B = [0, 1]$. Then $\overline{A \cap B} = [0, 1]$, which is not open.

(c) (Both). Sometimes. If we let $A = B = \mathbb{R}$, then we have $\overline{A \cap B} = \mathbb{R}$ which is both open and closed. On the other hand, if we let $A = B = [0, 1]$. Then $\overline{A \cap B} = [0, 1]$, which is not open, and hence not both open and closed.

(d) (Neither). Never. Since $\overline{A \cap B}$ is always closed, it can never be neither open nor closed.

(e) (Open). Always. Note that we can write $A \setminus B = A \cap B^c$. Therefore if $A$ is open and $B$ is closed, $B^c$ is open, and therefore this is the intersection of finitely many open sets, which is open.

(f) (Closed). Sometimes. If we let $B = \emptyset$ and $A = \mathbb{R}$, then $A \setminus B = \mathbb{R}$, which is closed. On the other hand, if we let $A = (0, 2)$ and $B = [1, 3]$, then $A \setminus B = (0, 1)$ which is not closed.

(g) (Both). Sometimes. If we let $B = \emptyset$ and $A = \mathbb{R}$, then $A \setminus B = \mathbb{R}$, which is both open and closed. On the other hand, if we let $A = (0, 2)$ and $B = [1, 3]$, then $A \setminus B = (0, 1)$ which is not closed, and hence not both open and closed.

(h) (Neither). Never. Since $A \setminus B$ is open, it can never be neither open nor closed.

(i) (Closed). Always. Since $A$ is open, $A^c$ is closed. Since $B$ is open and the union of open sets is open, $A^c \cup B$ is open. Then $(A^c \cup B)^c$ is closed.

(j) (Open). Sometimes. If we let $A = B = \mathbb{R}$, then we have $(A^c \cup B)^c = \emptyset$ which is open. On the other hand, if we let $A = \mathbb{R}$, $B = (0, 1)$. Then $(A^c \cup B)^c = (-\infty, 0] \cup [1, \infty)$ which is not open.

(k) (Both). Sometimes. If we let $A = B = \mathbb{R}$, then we have $(A^c \cup B)^c = \emptyset$ which is both open and closed. On the other hand, if we let $A = \mathbb{R}$, $B = (0, 1)$. Then $(A^c \cup B)^c = (-\infty, 0] \cup [1, \infty)$ which is not open, and hence not both open and closed.

(l) (Neither). Never. Since $(A^c \cup B)^c$ is always closed, it can never be neither open nor closed.
(d) In this case, in fact we have that \((A \cap B) \cup (A^c \cap B) = B\). Therefore,

- (Open). Always. By hypothesis, \(B\) is open.
- (Closed). Sometimes. If \(B = \emptyset, \mathbb{R}\), then \(B\) is closed as well. If \(B = (0, 1)\), then \(B\) is not closed.
- (Both). Sometimes. If \(B = \emptyset, \mathbb{R}\), then \(B\) is open and closed. If \(B = (0, 1)\), then \(B\) is not both closed and open.
- (Neither). Never. Since \(B\) is always open, it follows that \(B\) is never neither open nor closed.

(e) In this case, since \(A\) is open, it follows that \(A^c\) is closed, and hence that \(A^c \supseteq \overline{A}\). Since \(A \subseteq \overline{A}\), we have \((A)^c \subseteq A^c\), hence \((A)^c \cap \overline{A} = (A)^c\).

- (Open). Always. Since \(A\) is closed, \((A)^c\) is open.
- (Closed). Sometimes. If \(A = \emptyset\), then \((A)^c = \mathbb{R}\) which is closed. If \(A = (0, 1)\), then \(A = [0, 1]\) and \((A)^c = (-\infty, 0) \cup (1, \infty)\) which is not closed.
- (Both). Sometimes. If \(A = \emptyset\), then \((A)^c = \mathbb{R}\) which is both closed and open. If \(A = (0, 1)\), then \(A = [0, 1]\) and \((A)^c = (-\infty, 0) \cup (1, \infty)\) which is not closed and therefore not both open and closed.
- (Neither). Never. Since \((A)^c\) is open is always open, it follows that \((A)^c\) is open is never neither open nor closed.

Exercise 8: (Abbot Exercise 3.2.14)

Define the interior of a set \(E\) to be the set

\[ E^\circ = \bigcup_{U \subseteq E \atop U \text{ open}} U \]

(a) Prove that \(E^\circ = \{x \in E \mid V_\epsilon(x) \subseteq E\text{ for some } \epsilon > 0\}\).

(b) Show that \(E\) is closed if and only if \(\overline{E} = E\). Show that \(E\) is open if and only if \(E^\circ = E\).

(c) Show that \((\overline{E})^c = (E^\circ)^c\) and \((E^\circ)^c = \overline{E}\).

Solution 8.

(a) Define \(A = \{x \in E \mid V_\epsilon(x) \subseteq E\text{ for some } \epsilon > 0\}\). We show mutual containment.

- (\(\subseteq\)). Given any open subset \(U \subseteq E\), for any \(x \in U\), by openness of \(U\), we have that there exists \(\epsilon > 0\) such that \(V_\epsilon(x) \subseteq U \subseteq E\). But this implies that \(x \in A\), hence \(U \subseteq A\). Since \(U\) was an arbitrary open subset of \(E\), it follows that \(E^\circ = \bigcup_{U \subseteq E \atop U \text{ open}} U \subseteq A\) as well.

- (\(\supseteq\)). We’ll show that \(A\) is an open set. Given any \(y \in A\), we have that \(y \in E\) and there exists some \(\epsilon > 0\) such that \(V_\epsilon(y) \subseteq E\). However, since the set \(V_\epsilon(y)\) is open, for any \(z \in V_\epsilon(y)\), there exists some \(\delta > 0\) such that \(V_\delta(z) \subseteq V_\epsilon(y) \subseteq E\). Therefore \(z \in A\) as well. Hence in fact we have \(V_\epsilon(y) \subseteq A\). Since \(y \in A\) was arbitrary, it follows that \(A\) is open. Note also that \(A \subseteq E\). So \(A\) is one of the sets in the union \(\bigcup_{U \subseteq E \atop U \text{ open}} U\), which implies that \(A \subseteq E^\circ\).

(b) (Closure). Recall that \(\overline{E} = E \cup L\) where \(L\) is the set of limit points of \(E\).

- (\(\Rightarrow\)). If \(E\) is closed, then \(E\) contains all of its limit points, therefore \(L \subseteq E\), so \(\overline{E} = E \cup L = E\).

- (\(\Leftarrow\)). If \(\overline{E} = E\), then because the closure of any set is closed, it follows that \(E\) is closed as well.

- (Interior).

- (\(\Rightarrow\)). If \(E\) is open, then for every point in \(x\) there exists \(\epsilon > 0\) such that \(V_\epsilon(x) \subseteq E\). Therefore every \(x\) is in \(\{x \in E \mid V_\epsilon(x) \text{ for some } \epsilon > 0\} = E^\circ\). Since it’s clear from the definition that \(E^\circ \subseteq E\), we have that \(E = E^\circ\).

- (\(\Leftarrow\)). If \(E^\circ = E\), then since \(E^\circ\) is by definition a union of open sets, it is itself open, and therefore \(E\) is open as well.

(c) \(((\overline{E})^c = (E^\circ)^c)\). We show mutual containment.

- (\(\subseteq\)). Since \(\overline{E}\) is a closed set, it follows that \((\overline{E})^c\) is an open set. Furthermore since \(E \subseteq \overline{E}\), we have that \((\overline{E})^c \subseteq E^c\). Therefore \((\overline{E})^c\) is an open set which is contained in \(E^c\), so it is contained in \((E^c)^c\).

- (\(\supseteq\)). Suppose that \(U\) is an open subset of \(E^c\). Then for any \(x \in U\), there exists some \(\epsilon > 0\) such that \(V_\epsilon(x) \subseteq U\). This shows that \(x\) is not in \(E\) and that \(x\) is not a limit point of \(E\), and
Solution 9.

(a) 
- **(Closed).** Always. Since \( K \) is compact, it is closed, and since \( K \) and \( F \) are closed, \( K \cap F \) is closed.
- **(Open).** Sometimes. If \( K = [0, 1] \) and \( F = \emptyset \), then \( K \cap F = \emptyset \) which is open. However let \( K = [0, 1] = F \). Then \( K \cap F = [0, 1] \) which is not open.
- **(Both).** Sometimes. If \( K = [0, 1] \) and \( F = \emptyset \), then \( K \cap F = \emptyset \) which is both open and closed. However let \( K = [0, 1] = F \). Then \( K \cap F = [0, 1] \) which is not both open and closed.
- **(Neither).** Never. Since \( K \cap F \) is always closed, it follows that \( K \cap F \) is never neither open nor closed.

(b) 
- **(Closed).** Always. The closure of any set is a closed set.
- **(Open).** Sometimes. Let \( F = K = \emptyset \). Then \( F^c = K^c = \mathbb{R} \), hence \( F^c \cup K^c = \mathbb{R} \), so \( F^c \cup K^c = \mathbb{R} \), which is open. But if we let \( F = K = [0, 1] \),
- **(Both).** Sometimes. Let \( F = K = \emptyset \). Then \( F^c = K^c = \mathbb{R} \), hence \( F^c \cup K^c = \mathbb{R} \), so \( F^c \cup K^c = \mathbb{R} \), which is also open.
- **(Neither).** Never. Since \( F^c \cup K^c \) is always closed, it follows that \( F^c \cup K^c \) is never neither open nor closed.

(c) 
- **(Closed).** Sometimes. If \( K = F = \emptyset \), then \( K \setminus F = \emptyset \), which is closed. On the other hand, if \( K = [0, 3] \) and \( F = [1, 2] \), then \( K \setminus F = [0, 1] \cup [2, 3] \), which is not closed.
- **(Open).** Sometimes. If \( K = F = \emptyset \), then \( K \setminus F = \emptyset \), which is open. On the other hand, if \( K = [0, 3] \) and \( F = [1, 2] \), then \( K \setminus F = [0, 1] \cup [2, 3] \), which is not open.
- **(Both).** Sometimes. If \( K = F = \emptyset \), then \( K \setminus F = \emptyset \), which is both open and closed. On the other hand, if \( K = [0, 3] \) and \( F = [1, 2] \), then \( K \setminus F = [0, 1] \cup [2, 3] \), which is not both open and closed.
- **(Neither).** Sometimes. If \( K = F = \emptyset \), then \( K \setminus F = \emptyset \), which is not neither. On the other hand, if \( K = [0, 3] \) and \( F = [1, 2] \), then \( K \setminus F = [0, 1] \cup [2, 3] \), which is neither open nor closed.

(d) 
- **(Closed).** Always. The closure of any set is always closed.
- **(Open).** Sometimes. If \( K = \emptyset, F = \mathbb{R} \), then \( K \cap F^c = \emptyset \), which is open. But if \( K = [0, 1] \) and \( F = \emptyset \), then \( K \cap F^c = [0, 1] \), which is not open.
- **(Both).** Sometimes. If \( K = \emptyset, F = \mathbb{R} \), then \( K \cap F^c = \emptyset \), which is both open and closed. But if \( K = [0, 1] \) and \( F = \emptyset \), then \( K \cap F^c = [0, 1] \), which is not both open and closed.
- **(Neither).** Never. Since it is always closed, it is never neither open nor closed.
Exercise 10: (Abbot Exercise 3.3.4)

Decide whether the following propositions are true or false. If they are true, prove it. If not, find a counter example.

(a) The arbitrary intersection of compact sets is compact.
(b) The arbitrary union of compact sets is compact.
(c) Let $A$ be arbitrary and let $K$ be compact. Then, the intersection $A \cap K$ is compact.
(d) If $F_1 \supseteq F_2 \supseteq \ldots$ is a nested sequence of non-empty closed sets, then we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution 10.

(a) This is true. Let $\{K_{\alpha} \mid \alpha \in I\}$ be a collection of compact sets. Then by the Heine-Borel theorem, each $K_{\alpha}$ is closed and bounded. Since $\bigcap_{\alpha \in I} K_{\alpha}$ is contained in every $K_{\alpha}$, it follows that this intersection is bounded as well. Furthermore, because each $K_{\alpha}$ is closed and the arbitrary intersection of closed sets is closed, it follows that $\bigcap_{\alpha \in I} K_{\alpha}$ is closed as well. Hence $\bigcap_{\alpha \in I} K_{\alpha}$ is closed and bounded, which implies it is compact by the Heine-Borel theorem.

(b) This is false. For $n \in \mathbb{Z}$, let $K_n = [n, n+1]$. Then each $K_n$ is closed and bounded, so by the Heine-Borel theorem, each $K_n$ is compact. However, we compute

$$\bigcup_{n=1}^{\infty} K_n = \mathbb{R},$$

which is not bounded, and therefore not compact by the Heine-Borel theorem.

(c) This is false. The interval $K = [0, 2]$ is closed and bounded and hence compact, yet if we let $A = (1, 3)$, then $K \cap A = [1, 2]$ which is not closed, and therefore not compact.

(d) This is false. (And actually the same as Exercise 5 (b)). Let $F_1 = \mathbb{N}$, $F_2 = \mathbb{N} \setminus \{1\}$, $F_3 = \mathbb{N} \setminus \{1, 2\}$, and for any $n \in \mathbb{N}$, $F_n = \mathbb{N} \setminus \{1, 2, \ldots, n-1\}$. A similar proof to that which we used to show that $\mathbb{N}$ is closed shows that $F_n$ is closed for all $n \in \mathbb{N}$. Furthermore $n \in F_n$, so $F_n$ is nonempty. Finally, we have that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$$

However for each $n \in \mathbb{N}$, we have that $n \notin F_{n+1}$, and therefore $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

2. Practice Problems (Don’t hand these in)

Exercise 11: (Abbot Exercise 3.2.11)

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(b) Does this result about closures extend to infinite unions of sets?