1. Algebraic Limit Theorem

**Theorem 1: Algebraic Limit Theorem (Abbott Theorem 2.3.3)**

Let \((a_n)\) and \((b_n)\) be sequences of real numbers such that \(\lim_{n \to \infty} a_n = a\) and \(\lim_{n \to \infty} b_n = b\). Then the following statements hold.

(a) \(\lim_{n \to \infty} (ca_n) = ca\).
(b) \(\lim_{n \to \infty} (a_n + b_n) = a + b\).
(c) \(\lim_{n \to \infty} (a_n b_n) = ab\).
(d) If \(b_n \neq 0\) for all \(n\) and \(b \neq 0\), then \(\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}\).

**Proof.**

(a) We consider two cases.

- *(Scratch Work).* We are trying to show that \((ca_n) \to ca\). Hence we are interested in the inequality

\[
|ca_n - ca| < \varepsilon
\]

\[
\iff |c(a_n - a)| < \varepsilon
\]

\[
\iff |c||a_n - a| < \varepsilon
\]

\[
\iff |a_n - a| < \frac{\varepsilon}{|c|}.
\]

- *(Proof).* Let \(\varepsilon > 0\) be arbitrary. Since \((a_n)\) converges to \(a\), given \(\frac{\varepsilon}{|c|} > 0\), there exists \(N \in \mathbb{N}\) such that if \(n \geq N\), then \(|a_n - a| < \frac{\varepsilon}{|c|}\). Then we compute

\[
|a_n - a| < \frac{\varepsilon}{|c|}
\]

\[
\implies |c||a_n - a| < \varepsilon
\]

\[
\implies |c(a_n - a)| < \varepsilon
\]

\[
\implies |ca_n - ca| < \varepsilon
\]

Therefore for any \(\varepsilon > 0\), we have found \(N \in \mathbb{N}\) such that if \(n \geq N\), then \(|ca_n - ca| < \varepsilon\). Hence \((ca_n)\) converges to \((ca)\).

(b) *(Scratch Work).* We are trying to show that \((a_n + b_n) \to a + b\). Hence we are interested in the quantity

\[
|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|
\]

\[
\overset{(1)}{\leq} |a_n - a| + |b_n - b|
\]

Here (1) follows from the triangle inequality (??). But since \((a_n) \to a\) and \((b_n) \to b\), we can make \(|a_n - a|\) and \(|b_n - b|\) as small as we want.

- *(Proof).* Let \(\varepsilon > 0\) be arbitrary. Since \((a_n) \to a\), there exists some \(N_1 \in \mathbb{N}\) such that if \(n \geq N_1\), we have that \(|a_n - a| < \frac{\varepsilon}{2}\). Similarly, since \((b_n) \to b\), there exists some \(N_2 \in \mathbb{N}\) such that if \(n \geq N_2\), we have that \(|b_n - b| < \frac{\varepsilon}{2}\). Let \(N = \max\{N_1, N_2\}\). Then we have that \(N \geq N_1\) and \(N \geq N_2\), so if \(n \geq N\), then \(n \geq N_1\) and \(n \geq N_2\), so we have both \(|a_n - a| < \frac{\varepsilon}{2}\) and \(|b_n - b| < \frac{\varepsilon}{2}\). Compute
First we show that if \( b \neq 0 \), then for any \( \epsilon > 0 \), we have found \( N \in \mathbb{N} \) such that if \( n \geq N \), then \( |(a_n + b_n) - (a + b)| < \epsilon \).

Therefore for any \( \epsilon > 0 \), we have found \( N \in \mathbb{N} \) such that if \( n \geq N \), then \( |(a_n + b_n) - (a + b)| < \epsilon \).

Hence \((a_n + b_n)\) converges to \((a + b)\).

(c) Scratch Work. We are trying to show that \((a_n b_n) \to ab\). Hence we are interested in the quantity

\[
|a_n b_n - ab| = |a_n b_n - ab + ab - ab| \\
\leq |a_n b_n - ab| + |ab - ab| \\
= |b_n| |a_n - a| + |a| |b_n - b|.
\]

Here (1) follows from the triangle inequality. Since \((a_n) \to a\) and \((b_n) \to b\), we can make \(|a_n - a|\) and \(|b_n - b|\) small. The number \(|a|\) is fixed, and since \((b_n)\) is a convergent sequence, \( |a| \) implies that \((b_n)\) is bounded. Hence we have some bound \( M \in \mathbb{R} \) with \(|b_n| \leq M\) for all \( n \in \mathbb{N}\).

(Proof). Let \( \epsilon > 0 \) be arbitrary. Since \((b_n)\) is a convergent sequence, \( M \) implies that \((b_n)\) is bounded. Hence we have some bound \( M \in \mathbb{R} \), which we can choose to satisfy \( M > 0 \), with \(|b_n| \leq M\) for all \( n \in \mathbb{N}\). Now we consider two cases.

- \((a = 0)\). Since \((a_n) \to a\), there exists some \( N \in \mathbb{N} \) such that if \( n \geq N \), we have \(|a_n - a| < \frac{\epsilon}{M}\). Then compute

\[
|a_n b_n - ab| = |a_n b_n - ab + ab - ab| \\
\leq |a_n b_n - ab| + |ab - ab| \\
= |b_n| |a_n - a| + |a| |b_n - b| \\
< M \frac{\epsilon}{M} + 0 \\
= \epsilon.
\]

- \((a \neq 0)\). Since \((a_n) \to a\), there exists some \( N_1 \in \mathbb{N} \) such that if \( n \geq N_1 \), we have \(|a_n - a| < \frac{\epsilon}{2M}\). Since \((b_n) \to b\) and \( a \neq 0 \), there exists some \( N_2 \in \mathbb{N} \) such that if \( n \geq N_2 \), we have \(|b_n - b| < \frac{\epsilon}{2|a|}\). Let \( N = \max\{N_1, N_2\}\). Then we have that \( N \geq N_1 \) and \( N \geq N_2 \), so if \( n \geq N \), then \( n \geq N_1 \) and \( n \geq N_2 \), so we have both \(|a_n - a| < \frac{\epsilon}{2M}\) and \(|b_n - b| < \frac{\epsilon}{2|a|}\). Compute

\[
|a_n b_n - ab| = |a_n b_n - ab + ab - ab| \\
\leq |a_n b_n - ab| + |ab - ab| \\
= |b_n| |a_n - a| + |a| |b_n - b| \\
< M \frac{\epsilon}{2M} + |a| \frac{\epsilon}{2|a|} \\
= \epsilon.
\]

Therefore in either case for any \( \epsilon > 0 \), we have found \( N \in \mathbb{N} \) such that if \( n \geq N \), then \(|(a_n b_n) - (ab)| < \epsilon\). Hence \((a_n b_n)\) converges to \((ab)\).

(d) First we show that if \( b_n \neq 0 \) for all \( n \in \mathbb{N} \), then \( \left( \frac{1}{b_n} \right) \to \frac{1}{b} \).

(Scratch Work). We are trying to show that \( \left( \frac{1}{b_n} \right) \to \frac{1}{b} \). Hence we are interested in the quantity

\[
\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b| |b_n|}.
\]

We need to make the quantity \( \frac{|b - b_n|}{|b| |b_n|} \) small. Since \((b_n) \to b\), and \(|b|\) is a fixed number, we can make \( \frac{|b - b_n|}{|b_n|} \) small, but to make \( \frac{1}{|b_n|} \) small, we need a lower bound on the terms \( b_n \). This is make possible by the fact that \((b_n) \to b\) and \( b \) is nonzero, hence the terms of \( b_n \) are eventually bounded away from 0.
\begin{itemize}
  \item \textbf{(Proof).} Let $\epsilon > 0$ be arbitrary. Since $(b_n) \to b$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|b_n - b| < \frac{|b|}{2}$. Using the reverse triangle inequality \eqref{reversetriangle}, we obtain that

  \[
  |b_n| - |b| \leq |b_n - b| < \frac{|b|}{2}
  \]

  \[
  \implies -\frac{|b|}{2} < |b_n| - |b|
  \]

  \[
  \implies 0 < \frac{|b|}{2} < |b_n|
  \]

  \[
  \implies (1) \quad \frac{1}{|b_n|} < \frac{2}{|b|}.
  \]

  Here (1) follows from \eqref{reversetriangle} part (e).

  Furthermore, because $(b_n) \to b$, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\epsilon |b|^2}{2}$.

  Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$, we have that

  \[
  \left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b||b_n|} < \frac{\epsilon |b|^2}{2} \cdot \frac{1}{|b|} \cdot \frac{2}{|b|} = \epsilon.
  \]

  Therefore for any $\epsilon > 0$, we have found $N \in \mathbb{N}$ such that if $n \geq N$, then $\left| \frac{1}{b_n} - \frac{1}{b} \right| < \epsilon$. Hence \( \left( \frac{1}{b_n} \right) \) converges to $\frac{1}{b}$.

  Since $(a_n) \to a$ and \( \left( \frac{1}{b_n} \right) \to \frac{1}{b} \), it follows from part (c) that \( \left( \frac{a_n}{b_n} \right) \to \frac{a}{b} \), as desired.

  \end{itemize}

\section*{Exercise 1: Abbott Exercise 2.3.1}

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

\section*{Exercise 2: Abbott Exercise 2.3.4}

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem \eqref{algebraiclimit} to compute the following limits, and prove your result. (Assume that the sequence $(a_n)$ is such that all fractions below are defined).

(a) $\lim_{n \to \infty} \frac{1+2a_n}{1+3a_n-4a_n^2}$

(b) $\lim_{n \to \infty} \frac{(a_n+2)^2-4}{a_n}$

(c) $\lim_{n \to \infty} \frac{2+3}{\frac{a_n^3}{a_n+5}}$

\section*{Exercise 3: Abbott Exercise 2.3.7}

Give an example of each of the following, or prove that such a request is impossible.

(a) Sequences $(x_n)$ and $(y_n)$, which both diverge, but whose sum $(x_n + y_n)$ converges.

(b) Sequences $(x_n)$ and $(y_n)$, where $(x_n)$ converges, $(y_n)$ diverges, and $(x_n + y_n)$ converges.

(c) A convergent sequence $(b_n)$ with $b_n \neq 0$ for all $n \in \mathbb{N}$ such that $(1/b_n)$ diverges.

(d) An unbounded sequence $(a_n)$ and a convergent sequence $(b_n)$ with $(a_n - b_n)$ bounded.

(c) Two sequences $(a_n)$ and $(b_n)$, where $(a_nb_n)$ and $(a_n)$ converge but $(b_n)$ does not.
Exercise 4: Abbott Exercise 2.3.9

(a) Let \((a_n)\) be a bounded (not necessarily convergent) sequence, and assume that \(\lim_{n \to \infty} b_n = 0\). Show that \(\lim_{n \to \infty} (a_n b_n) = 0\). Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of \((a_n b_n)\) if we assume that \((b_n)\) converges to some nonzero limit \(b\)?

2. Order Limit Theorem

**Theorem 2: Order Limit Theorem (Abbott Theorem 2.3.4)**

Let \((a_n)\) and \((b_n)\) be sequences of real numbers such that \(\lim_{n \to \infty} a_n = a\) and \(\lim_{n \to \infty} b_n = b\). Then the following statements hold.

(a) If \(a_n \geq 0\) for all \(n \in \mathbb{N}\), then \(a \geq 0\).

(b) If \(a_n \leq b_n\) for all \(n \in \mathbb{N}\), then \(a \leq b\).

(c) If there exists \(c \in \mathbb{R}\) for which \(c \leq b_n\) for all \(n\), then \(c \leq b\). Similarly, if \(a_n \leq c\) for all \(n \in \mathbb{N}\), then \(a \leq c\).

**Proof.**

(a) Suppose, for contradiction, that \(a < 0\). Then let \(\epsilon = \frac{-a}{2}\), and note that \(\epsilon > 0\). Since \((a_n) \to a\), it follows that there exists \(N \in \mathbb{N}\) such that if \(n \geq N\), then \(|a_n - a| < \epsilon\). Since \(N \geq N\), we have that

\[
|a_N - a| < \epsilon
\]

\[
\implies |a_N - a| < \frac{-a}{2}
\]

\[
\implies a_N - a < \frac{-a}{2}
\]

\[
\tag{1}
\implies a_N < \frac{a}{2}
\]

Here (1) follows by adding \(a\) to both sides of the inequality. However \(a < 0\), hence dividing both sides by 2 yields \(\frac{a}{2} < 0\). Therefore we have that \(a_N < \frac{a}{2} < 0\), contradicting that \(a_n \geq 0\) for all \(n \in \mathbb{N}\).

(b) Since \((a_n) \to a\), part (a) implies that \((-a_n) \to -a\), and therefore part (b) implies that \((b_n - a_n) \to b - a\). Since \(a_n \leq b_n\) for all \(n \in \mathbb{N}\), it follows that \(0 \leq b_n - a_n\) for all \(n \in \mathbb{N}\). Therefore applying part (a) of this theorem then yields that \(0 \leq b - a\), which implies that \(a \leq b\), as desired.

(c) Let \((x_n)\) be the sequence defined by \(x_n = x\) for all \(n \in \mathbb{N}\). Then we have that \(x_n \leq b_n\) for all \(n\). By part (b), \((x_n) \to c\), hence applying part (b) of this proposition yields that \(c \leq b\). A similar argument shows that \(a \leq c\) if \(a_n \leq c\) for all \(n \in \mathbb{N}\).

3. Eventuality

**Definition 1: Eventually**

Let \((a_n)\) be a sequence for which some property of \(a_n\) holds for all \(n \geq N\) for some \(N \in \mathbb{N}\). Then we say that \((a_n)\) *eventually* has this property.
Note 1: Eventuality

Properties of the limit of a convergent sequence, in general, do not depend on the behavior of the beginning of the sequence. In this sense, only eventuality matters for the limit.

Example 1: Eventuality

(a) Let \( c \in \mathbb{R} \) be a real number. A sequence \((a_n)\) is eventually \( \geq c \) if there exists some \( N \in \mathbb{N} \) such that \( a_n \geq 0 \) for all \( n \geq N \).

(b) Given two sequences \((a_n)\) and \((b_n)\), we say that \((a_n)\) is eventually less than or equal to \((b_n)\), if there exists some \( N \in \mathbb{N} \) such that \( a_n \leq b_n \) for all \( n \geq N \).

Exercise 5: Eventually

Show that the Order Limit Theorem (??) still holds if the conditions are relaxed to being only eventually true.

Proof. Homework.

4. Monotone Convergence Theorem

Definition 2: Increasing, Decreasing, and Monotone

A sequence \((a_n)\) is increasing if \( a_n \leq a_{n+1} \) for all \( n \in \mathbb{N} \) and decreasing if \( a_n \geq a_{n+1} \) for all \( n \in \mathbb{N} \). A sequence is monotone if it is either increasing or decreasing.

Theorem 3: Monotone Convergence Theorem (Abbott Theorem 2.4.2)

If a sequence is monotone and bounded, then it converges.