1. Monotone Convergence Theorem

Definition 1: Increasing, Decreasing, and Monotone

A sequence \((a_n)\) is **increasing** if \(a_n \leq a_{n+1}\) for all \(n \in \mathbb{N}\) and **decreasing** if \(a_n \geq a_{n+1}\) for all \(n \in \mathbb{N}\). A sequence is **monotone** if it is either increasing or decreasing.

Theorem 1: Monotone Convergence Theorem (Abbott Theorem 2.4.2)

If a sequence is monotone and bounded, then it converges.

**Proof.** We consider two cases.

- \((a_n \text{ is increasing})\). Since \((a_n)\) is bounded, ?? implies that the set \(A = \{a_n \mid n \in \mathbb{N}\}\) is a bounded subset of \(\mathbb{R}\). Hence the Least Upper Bound Property of \(\mathbb{R}\) implies that \(A\) has a supremum. Let \(s = \sup A\). We will show that \((a_n) \to s\). Given \(\epsilon > 0\), since \(s = \sup A\), there exists some element \(a_N \in A\) such that \(s - \epsilon < a_N\). Given any \(n \geq N\), since \((a_n)\) is increasing, we have that \(a_N \leq a_n\), and therefore \(s - \epsilon < a_n\) as well. But since \(a_n \in A\), it follows that \(a_n \leq s < s + \epsilon\). Therefore \(s - \epsilon < a_n < s + \epsilon\), and hence \(|a_n - s| < \epsilon\) for all \(n \geq N\). Hence we have shown that for all \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that if \(n \geq N\), then \(|a_n - s| < \epsilon\), which implies that \((a_n) \to s\). Hence indeed \((a_n)\) converges.

- \((a_n \text{ is decreasing})\). Homework.

Exercise 1: MCT Decreasing Case

Prove the Monotone Convergence Theorem (??) for decreasing sequences.

**Proof.** Homework.

2. Infinite Series

Definition 2: Series

Let \((b_n)\) be a sequence of real numbers. An **infinite series** (or just **series**) is a formal expression of the form

\[
\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \ldots
\]

Definition 3: Sequence of Partial Sums

Given a series \(\sum_{n=1}^{\infty} b_n\) of real numbers, the **sequence of partial sums** of this series is the sequence \((s_m)\) where

\[
s_m = \sum_{n=1}^{m} b_n = b_1 + b_2 + \ldots + b_m.
\]
Definition 4: Convergence of a Series

A series \( \sum_{n=1}^{\infty} b_n \) converges to a real number \( B \) if its sequence of partial sums \( (s_m) \) converges (in the sense of ??) to \( B \). In this case we write \( \sum_{n=1}^{\infty} b_n = B \). If a series does not converge to any real number, we say it diverges.

Example 1: Convergence of Series (Abbott Example 2.4.4)

Consider the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

All of the terms in this series are positive, hence the sequence of partial sums \( (s_m) \) with

\[
s_m = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{m^2},
\]

is an increasing sequence. Furthermore, we compute

\[
s_m = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{m^2} \\
= 1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \ldots + \frac{1}{m} \cdot \frac{1}{m} \\
< 1 + \frac{1}{2} \cdot \frac{1}{1} + \frac{1}{3} \cdot \frac{1}{2} + \ldots + \frac{1}{m} \cdot \frac{1}{m-1} \\
= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
= 1 + 1 - \frac{1}{m} \\
< 2.
\]

This implies that the sequence \( (s_m) \) of partial sums is bounded. Since we already showed it was increasing, by the Monotone Convergence Theorem (??), \( (s_m) \) converges to some \( s \in \mathbb{R} \), hence the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges to \( s \). However at this point in the course we are not yet able to calculate \( s \).
Example 2: Harmonic Series (Abbott Example 2.4.5)

Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$ 

The sequence \((s_m)\) of partial sums of the harmonic series is given by

$$s_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}.$$ 

In particular, we list several terms of this sequence

- \(s_2 = 1 + \frac{1}{2}\)
- \(s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)\)
- \(> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)\)
- \(= 1 + \frac{1}{2} + \frac{1}{2}\)
- \(s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)\)
- \(> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)\)
- \(= 1 + \frac{1}{2} + \frac{1}{2}\)

In general, we compute

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{2^{k-1} + 1} + \ldots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{2^k + \ldots + \frac{1}{2^k}}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}$$

$$= 1 + k \left(\frac{1}{2}\right).$$

Therefore the sequence of partial sums \((s_m)\) is unbounded. Since convergent sequences are bounded (??), \((s_m)\) cannot be convergent.

3. Subsequences

Definition 5: Subsequence (Abbott Definition 2.5.1)

Let \((a_n)\) be a sequence of real numbers, and let \(n_1 < n_2 < \ldots\) be an increasing sequence of natural numbers. Then the sequence

\[(a_{n_1}, a_{n_2}, a_{n_3}, \ldots)\]

is called a subsequence of \((a_n)\) and is denoted by \((a_{n_k})\) where \(k \in \mathbb{N}\) indexes the subsequence.
Example 3: Subsequences

Consider the sequence \((a_n) = \left(\frac{1}{n}\right)_{n=1}^\infty\). One subsequence of this sequence is

\[(a_{n_k}) = (1, 1/3, 1/5, 1/7, \ldots)\]

where \(n_k = 2k - 1\).

Theorem 2: Convergence of Subsequences (Abbott Theorem 2.5.2)

Let \((a_n)\) be a sequence of real numbers which converges to \(a \in \mathbb{R}\). Then any subsequence \((a_{n_k})\) of \((a_n)\) also converges to \(a\).

Proof. Let \(\epsilon > 0\) be arbitrary. Because \((a_n) \to a\), there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\), we have \(|a_n - a| < \epsilon\). Note that \(n_k \geq k\) for all \(k \in \mathbb{N}\). Therefore if \(k \geq N\), we have \(n_k \geq k \geq N\), which implies that \(|a_{n_k} - a| < \epsilon\). Therefore \((a_{n_k}) \to a\) as well. \(\blacksquare\)

Example 4: Convergence of Subsequences (Abbott Example 2.5.3)

Let \(0 < b < 1\). Then multiplying both sides of the inequality \(b < 1\) by \(b^k\) yields \(b^{k+1} < b^k\) for all \(k \in \mathbb{N}\). Therefore, we have

\[b > b^2 > b^3 > \ldots\]

Therefore the sequence \((b^n)\) is decreasing, and bounded (by \(|b^k| < 1\) for all \(k \in \mathbb{N}\)), the Monotone Convergence Theorem (??) implies that \((b^n)\) converges to some limit \(\ell\). Consider the subsequence \((b^{2n})\). By ??, \((b^{2n})\) converges to the same limit \(\ell\) as \((b^n)\). But we have that \(b^{2n} = b^n \cdot b^n\). Therefore the algebraic limit theorem implies that

\[\ell = \lim_{n \to \infty} (b^n \cdot b^n) = \left(\lim_{n \to \infty} b^n\right) \cdot \left(\lim_{n \to \infty} b^n\right) = \ell^2.\]

This implies that either \(\ell = 0\) or \(\ell = 1\). However all terms of this sequence satisfy \(b^n \leq b\). Therefore by the Order Limit Theorem (??), \(\ell \leq b < 1\). Hence \(\ell = 0\).

Example 5: Using Subsequences to show their Parent Sequences Diverge (Abbott Example 2.5.4)

Recall the sequence \((a_n)\) given by \(a_n = (-1)^{n+1}\). Explicitly, this sequence is given by

\[(1, -1, 1, -1, -1, \ldots)\]

We can finally show that \((a_n)\) diverges. Suppose, for contradiction, that \((a_n) \to a\). Then we have the subsequence \((a_1, a_3, a_5, \ldots) = (1, 1, 1, \ldots)\). This is a constant sequence, and therefore converges to 1. By ??, it follows that \(a = 1\). Similarly, though, the subsequence \((a_2, a_4, a_6, \ldots) = (-1, -1, -1, \ldots)\) converges to \(-1\). Therefore \(a = -1\). Since we showed that limits are unique and \(1 \neq -1\), this contradicts our assumption that \((a_n)\) converges. Hence it diverges.

4. BOLZANO-WEIERSTRASS THEOREM

Theorem 3: Bolzano-Weierstrass Theorem (Abbott Theorem 2.5.5)

Every bounded sequence contains a convergent subsequence.

Proof. Let \((a_n)\) be a bounded sequence. Then there exists some \(M > 0\) such that \(|a_n| \leq M\) for all \(n \in \mathbb{N}\). This implies that \(a_n \in [-M, M]\) for all \(n \in \mathbb{N}\).
• We bisect the interval \([-M, M]\) into two closed intervals of equal length. One of these intervals must contain infinitely many terms of the sequence \((a_n)\). Let \(I_1\) be that interval, and let \(a_{n_1}\) be any point in \(I_1\).

• Next we bisect \(I_1\) into two closed intervals, and note that one of these intervals must contain infinitely many terms of the sequence \((a_n)\). Let \(I_2\) be that interval, and choose \(a_{n_2}\) inside this interval which satisfies \(n_2 \geq n_1\).

• In general, we bisect \(I_{k-1}\) into two closed intervals, one which must contain infinitely many terms of \((a_n)\). Let \(I_k\) be this closed interval, and choose \(a_{n_k} \in I_k\) such that \(n_k > n_{k-1}\).

Therefore we have obtained a subsequence \((a_{n_1}, a_{n_2}, \ldots)\) of \((a_n)\) and a sequence of nested intervals

\[ I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \]

By the Nested Interval Property (?), the intersection \(\bigcap_{n=1}^{\infty} I_n\) is nonempty, and therefore contains some element \(x\).

We claim that \((a_{n_k})\) converges to \(x\). Let \(\epsilon > 0\) be arbitrary. Then the length of the interval \(I_k\) is \(M \left(\frac{1}{2}\right)^{k-1}\). By ???, the sequence \(\left\{\frac{1}{2}\right\}^{k-1}\) converges to 0. Therefore, by the Algebraic Limit Theorem (?), the sequence \(M \left(\frac{1}{2}\right)^{k-1}\) converges to 0 as well. Hence, there exists some \(N \in \mathbb{N}\) such that if \(k \geq N\), then \(|M \left(\frac{1}{2}\right)^{k-1}| < \epsilon\).

But then \(a_{n_k}, x \in I_k\), hence \(|a_{n_k} - x| < \epsilon\). Therefore \((a_{n_k})\) converges to \(x\). ■