1. Limit Points

**Definition 1: Limit Point (Abbott Definition 3.2.4)**

Let $A$ be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is a limit point of $A$ if every $\varepsilon$-neighborhood $V_\varepsilon(x)$ of $x$ intersects $A$ at some point other than $x$, i.e. for all $\varepsilon > 0$, there exists some $y \neq x$ with $y \in V_\varepsilon(x) \cap A$.

**Note 1: Limit Points**

Notice that the definition of a limit point $x$ of $A$ does not say anything about whether or not $x \in A$. It could be that $x \in A$ or that $x \notin A$.

**Example 1: Limit Points**

(a) Let $c < d$. The set of limit points of $(c, d)$ is $[c, d]$.

(b) The set of limit points of $\mathbb{Q}$ is $\mathbb{R}$ since for any point $x \in \mathbb{R}$, and any $\varepsilon > 0$, there exists a rational number $r \in \mathbb{Q}$ satisfying $x < r < x + \varepsilon$ and hence $r \neq x$ and $r \in V_\varepsilon(x) \cap \mathbb{Q}$.

(c) A similar argument shows that the set of limit points of $\mathbb{I}$ is $\mathbb{R}$.

**Exercise 1: Limit Points**

Give an example of a set $A \subseteq \mathbb{R}$ and four points $x, y, z, w \in \mathbb{R}$ such that the following conditions are satisfied.

(a) $x$ is a limit point of $A$ and $x \in A$.

(b) $y$ is a limit point of $A$ but $y \notin A$.

(c) $z$ is not a limit point of $A$ and $z \in A$.

(d) $w$ is not a limit point of $A$ and $w \notin A$.

**Proof.** Let $A = (0, 2) \cup \{3\}$.

(a) $1$ is a limit point of $A$ and $1 \in A$.

(b) $0$ is a limit point of $A$ but $0 \notin A$.

(c) $3$ is not a limit point of $A$ and $3 \in A$.

(d) $-1$ is not a limit point of $A$ and $-1 \notin A$.

**Theorem 1: Properties of Limit Points (Abbott Theorem 3.2.5)**

A point $x \in \mathbb{R}$ is a limit point of a set $A$ if and only if there exists a sequence $(a_n)$ contained in $A$ with $a_n \neq x$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} a_n = x$.

**Proof.**

- $(\Rightarrow)$. Suppose that $x$ is a limit point of $A$. Then for each $n \in \mathbb{N}$, we have $1/n > 0$, hence there exists some point $a_n \neq x$ with $a_n \in V_{1/n}(x) \cap A$. Hence $(a_n)$ is a sequence with $a_n \in A$ and $a_n \neq x$ for all $n \in \mathbb{N}$. We claim that $(a_n) \to x$. Given $\varepsilon > 0$ arbitrary, by the Archimedean Property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ with $1/N < \varepsilon$. Then if $n \geq N$, we have
\[ a_n \in V_{1/n}(x) \implies |x - a_n| < \frac{1}{n} \leq \frac{1}{N} < \epsilon. \]

Hence indeed \((a_n) \to x\).

- \((\Leftarrow)\). Suppose that \((a_n)\) is a sequence of points in \(A\) with \(a_n \neq x\) for all \(n \in \mathbb{N}\), and \(\lim_{n \to \infty} a_n = x\). Then for all \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that if \(n \geq N\), we have \(|a_n - x| < \epsilon\). In particular, this implies that \(|a_N - x < \epsilon\), and hence \(a_N \in V_\epsilon(x) \cap A\). Hence we have shown that for all \(\epsilon > 0\), there exists some point \(a_N \in V_\epsilon(x) \cap A\) with \(a_N \neq x\), so \(x\) is a limit point of \(A\).

\[\square\]

**Definition 2: Isolated Point (Abbott Definition 3.2.6)**

A point \(a \in \mathbb{R}\) is called an isolated point of \(A\) if it is in \(A\) but is not a limit point of \(A\).

**Note 2: Isolated Point**

The point \(z\) from ?? is an isolated point of \(A\).

**Example 2: Isolated Points**

- (a) \(\mathbb{Q}\) has no isolated points since every element of \(\mathbb{R}\) is a limit point of \(\mathbb{Q}\), and therefore every element of \(\mathbb{Q}\) is an isolated point of \(\mathbb{Q}\).
- (b) A similar argument shows that \(I\) has no isolated points.

**Example 3: Sets of Sequences (Abbott Theorem 3.2.8)**

Let \(A = \{ \frac{1}{n} \mid n \in \mathbb{N}\}\).

- We claim that every point of \(A\) is isolated. For any \(1/n \in A\), let \(\epsilon = 1/n - 1/n + 1 > 0\). Then \(V_\epsilon(1/n) \cap A = \{1/n\}\). Hence there exists an \(\epsilon\)-neighborhood of \(1/n\) which only intersects \(A\) in \(\{1/n\}\). Therefore \(1/n\) is an isolated point for all \(n \in \mathbb{N}\). This implies that \(1/n\) is not a limit point for any \(n \in \mathbb{N}\).
- However, \(0\) is a limit point of \(A\). First note that since \((1/n) \to 0\), for any \(\epsilon > 0\), there exists some \(n \in \mathbb{N}\) such that \(1/n \in V_\epsilon(0)\). Hence \(0\) is a limit point of \(A\).

**Theorem 2: Limit Point Characterization of Closed Sets (Abbott Theorem 3.2.13)**

A set \(F \subseteq \mathbb{R}\) is closed if and only if it contains all of its limit points.

**Proof.**

- \((\Rightarrow)\). Suppose that \(F\) is closed and for contradiction, suppose that there exists \(x\) a limit point of \(F\) with \(x \notin F\). Then \(x \in \mathbb{R} \setminus F\), and since \(F\) is closed, \(\mathbb{R} \setminus F\) is open. Therefore there exists some \(\epsilon > 0\) such that \(V_\epsilon(x) \subseteq \mathbb{R} \setminus F\). Therefore, there exists some \(\epsilon > 0\) such that \(V_\epsilon(x) \cap F = \emptyset\), contradiction that \(x\) is a limit point of \(F\).
- \((\Leftarrow)\). Suppose that \(F\) contains all of its limit points. Then given any \(x \in \mathbb{R} \setminus F\), \(x\) is not a limit point of \(F\), hence there exists some \(\epsilon > 0\) such that \(V_\epsilon(x) \cap F = \emptyset\), which implies that \(V_\epsilon(x) \subseteq \mathbb{R} \setminus F\). Therefore \(\mathbb{R} \setminus F\) is open, so by definition, \(F\) is closed.

\[\square\]
Example 4: Limit Point Characterization of Closed Sets

(a) ?? gives another proof that Q and I are not closed: Since the set of limit points of Q is R, and yet Q ⊆ R, Q is not closed. The same is true for I.

(b) The set (c, d) is not closed, since c and d are both limit points of this set, and yet neither is in (c, d).

(c) The set A = \{1/n | n ∈ N\} is not closed, since 0 is a limit point, yet 0 \∉ A.

Theorem 3: Cauchy Sequence Characterization of Closed Sets (Abbott Theorem 3.2.8)

A set F ⊆ R is closed if and only if every Cauchy sequence in F has a limit that is also in F.

Proof.

- (⇒). Suppose that F ⊆ R is closed and let \((a_n)\) be a Cauchy sequence in F. By the Cauchy Criterion, \((a_n)\) converges to some point \(x ∈ R\). Then by ??, \(x\) is a limit point of F, hence by ??, \(x \in F\), as desired.

- (⇐). Suppose that every Cauchy sequence in F has a limit that is also in F. Let \(x ∈ R\) be a limit point of F. Then by ??, there exists some sequence \((a_n)\) in F which converges to \(x\). However convergent sequences are Cauchy, therefore \((a_n)\) is a Cauchy sequence contained in F, so by hypothesis, its limit \(x\) is also in F. Since \(x\) was an arbitrary limit point of \(F\), it follows that F contains all of its limit points, and therefore by ??, F is closed.

Note 3: Convergent Sequence Characterization of Closed Sets

Since, in R, a sequence is Cauchy if and only if it is convergent, we can restate ?? to say: F is closed if and only if every convergent sequence in F has a limit that is also in F.

Theorem 4: Density of Q in R (Abbott Theorem 3.2.10)

For every \(y ∈ R\), there exists a sequence of rational numbers that converges to \(y\).

Proof. Let \(y ∈ R\) and let \(ε > 0\) be arbitrary. Then the density of Q in R theorem states that there exists \(r ∈ Q\) such that \(y < r < y + ε\). Hence \(r ∈ V_{ε}(y)\), so \(y\) is a limit point of Q. By ??, there exists a sequence in Q which converges to \(y\).