1. Properties of $\mathbb{Q}$ (continued)

**Theorem 1: Archimedean Property of $\mathbb{Q}$ (Abbott Theorem 1.4.2)**

(a) Given any rational number $x \in \mathbb{Q}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
(b) Given any rational number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

**Proof.**

(a) Let $x = \frac{p}{q} \in \mathbb{Q}$. We consider two cases

- $(p/q \leq 0)$. Recall that ?? part (d) implies that $0 < 1$ in every ordered field, and therefore the inequality holds in $\mathbb{Q}$. Hence if $p/q \leq 0$, then by transitivity of the order relation, we have that $\frac{p}{q} < 1$, which gives the desired result.
- $(p/q > 0)$. Since $\frac{p}{q}$ is a positive rational number, there exists $r, s \in \mathbb{N}$ with $\frac{p}{q} = \frac{r}{s}$. Then compute

$$
(r + 1) - \frac{r}{s} = \frac{(r + 1)s - r}{s} = \frac{rs + s - r}{s} = \frac{r(s - 1) + s}{s}.
$$

Since $r, s \in \mathbb{N}$, $r(s - 1) + s > 0$ in $\mathbb{Z}$, and hence $\frac{r(s-1)+s}{s}$ is a positive rational number. Therefore

$$
\frac{r}{s} < r + 1,
$$

as desired.

(b) Since $y > 0$, by the proof of part (e) of ??, we have that $y^{-1} > 0$ as well. Then by part (a) of this proposition, there exists some $n \in \mathbb{N}$ with $y^{-1} < n$. Hence we have $0 < y^{-1} < n$. However, by ??, this implies that $0 < n^{-1} < (y^{-1})^{-1}$. ?? part (d) implies that $(y^{-1})^{-1} = y$. Furthermore, ?? implies that $n^{-1} = 1/n$, hence we obtain the desired result: there exists $n \in \mathbb{N}$ with $0 < 1/n < y$.


2. Supremums and Infimums

**Definition 1: Upper and Lower Bounds**

Let $(F, +, \times, <)$ be an ordered field, and let $A \subseteq F$ be a subset. An element $u \in F$ is said to be an upper bound for $A$ if $a \leq u$ for all $a \in A$. If $A \subseteq F$ has an upper bound, then it is said to be bounded above. Similarly $A$ is bounded below if there exists a lower bound $\ell \in \mathbb{R}$ satisfying $\ell \leq a$ for all $a \in A$.

**Definition 2: Bounded Set**

Let $(F, +, \times, <)$ be an ordered field. A subset $A \subseteq F$ is bounded if it is both bounded above and bounded below.
Definition 3: Supremum/Least Upper Bound

Let \((F, +, \times, <)\) be an ordered field, and let \(A \subseteq F\) be a subset. An element \(s \in F\) is said to be a supremum of \(A\) or a least upper bound of \(A\) (the two terms are used interchangeably) if the following two axioms hold.

(S1) \(s\) is an upper bound for \(A\).
(S2) If \(u\) is any other upper bound for \(A\), then \(s \leq u\).

Exercise 1: Uniqueness of a Supremum

Let \((F, +, \times, <)\) be an ordered field, and let \(A \subseteq F\) be a subset. Show that if \(r \in F\) and \(s \in F\) are both suprema of \(A\), then \(r = s\).

Proof. Suppose that \(r, s \in F\) are both suprema of \(A \subseteq F\). We'll show that \(r \leq s\) and \(s \geq r\), which will prove, by ??, that \(r = s\).

- \((\leq)\). Applying (S1) to \(s\), we see that, \(s\) is an upper bound for \(A\). However applying (S2) to \(r\), we see that since \(s\) is an upper bound for \(A\), we must have \(r \leq s\).
- \((\geq)\). This is the same as the previous part with \(r\) and \(s\) reversed: Applying (S1) to \(r\), we see that, \(r\) is an upper bound for \(A\). However applying (S2) to \(s\), we see that since \(r\) is an upper bound for \(A\), we must have \(s \leq r\), i.e. \(r \geq s\).

Note 1: Notation

Since suprema are unique, if \(A \subseteq F\) has a supremum, we will often denote it by \(\text{sup } A\).

Definition 4: Infimum/Greatest Lower Bound

Let \((F, +, \times, <)\) be an ordered field, and let \(A \subseteq F\) be a subset. An element \(t \in F\) is said to be an infimum of \(A\) or a greatest lower bound of \(A\) (the two terms are used interchangeably) if the following two axioms hold.

(I1) \(t\) is a lower bound for \(A\).
(I2) If \(\ell\) is any other lower bound for \(A\), then \(\ell \leq t\).

Exercise 2: Uniqueness of an Infimum

Let \((F, +, \times, <)\) be an ordered field, and let \(A \subseteq F\) be a subset. Show that if \(r \in F\) and \(t \in F\) are both infima of \(A\), then \(r = t\).

Proof. Homework.

Note 2: Notation

Since infima are unique, if \(A \subseteq F\) has an infimum, we will often denote it by \(\text{inf } A\).
Example 1: Supremums and Infimums in \( \mathbb{Q} \)

Recall that \( \mathbb{Q} \) is an ordered field. Consider the subset
\[
A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\} \subseteq \mathbb{Q}.
\]

Note that \( A \) is bounded above by 1 and bounded below by 0. We claim that \( \text{sup} \ A = 1 \) and \( \text{inf} \ A = 0 \).

- (sup \( A = 1 \)).
  - (S1) First note that \( 1 \geq \frac{1}{n} \) for all \( n \in \mathbb{N} \). Hence 1 is an upper bound for \( A \).
  - (S2) Suppose that \( b \in \mathbb{Q} \) is another upper bound for \( A \). Then for all \( a \in A \), we have that \( a \leq b \).
    Since 1 is actually in \( A \), we have that \( 1 \leq b \).

- (inf \( A = 0 \)).
  - (I1) First note that \( 0 \leq \frac{1}{n} \) for all \( n \in \mathbb{N} \). Therefore 0 is a lower bound for \( A \).
  - (I2) To show that 0 is the greatest lower bound for \( A \), we use proof by contradiction. Suppose there exists some lower bound \( \ell \in \mathbb{Q} \) for \( A \) such that \( 0 < \ell \). Since \( \ell \neq 0 \), by Theorem\[ \] part (b), there exists some \( N \in \mathbb{N} \) such that \( 0 < \frac{1}{n} < \ell \). But \( \frac{1}{n} \) is in \( A \), yet \( \ell \) is strictly larger than \( \frac{1}{n} \). This contradicts the assumption that \( \ell \) is a lower bound for \( A \). Therefore our assumption must have been false, i.e. there does not exist a lower bound for \( A \) with \( 0 < \ell \). Therefore 0 is greater than or equal to all lower bounds for \( A \).

3. The Least Upper Bound Property

**Definition 5: Least Upper Bound Property**

An ordered field \((F, +, \times, <)\) is said to have the least upper bound property if every set which is bounded above, has a supremum.

**Note 3: Thinking About the Least Upper Bound Property**

One way to rephrase Definition\[ \] is that an ordered field \( F \) has the least upper bound property the following condition holds: If a subset \( A \subseteq F \) has an upper bound, then it has a least upper bound.

**Exercise 3: Relationship between Supremums and Infimums (Abbott Exercise 1.3.3)**

Let \( F \) be an ordered field with the least upper bound property, and let \( A \subseteq F \) be nonempty and bounded below, and define \( B = \{ b \in F : b \) is a lower bound for \( A \} \). Show that \( \text{sup} \ B = \text{inf} \ A \).

**Proof.** Homework.

**Exercise 4: Supremums of Unions (Abbott Exercise 1.3.4)**

Let \( F \) be an ordered field with the least upper bound property and let \( A_1, A_2, \ldots, A_k \) be a collection of nonempty subsets of \( F \), each of which is bounded above. Find a formula for \( \text{sup} (A_1 \cup A_2) \). Extend this to a formula for \( \text{sup} (\bigcup_{k=1}^{n} A_k) \).

**Proof.** Homework.
### Exercise 5: Algebraic Properties of Supremums (Abbott Exercise 1.3.5 and 1.3.6)

Let \( F \) be an ordered field with the least upper bound property and let \( A, B \subseteq F \) be nonempty subsets which are bounded above. Let \( c \in F \), and define \( c + A := \{ c + a : a \in A \} \), \( cA := \{ ca : a \in A \} \), and \( A + B = \{ a + b : a \in A, b \in B \} \).

(a) Show that \( \sup(c + A) = c + \sup(A) \).

(b) Show that if \( c \geq 0 \) then \( \sup(cA) = c \sup(A) \).

(c) Show that \( \sup(A + B) = \sup(A) + \sup(B) \).

**Proof.**

(a) We will show that \( \sup(c + A) \leq c + \sup(A) \) and that \( \sup(c + A) \geq c + \sup(A) \). Then by ??, we will obtain the desired equality.

- \( (\leq) \). Given any element \( c + a \) of \( c + A \), Definition 3 axiom (S1) implies that \( \sup(A) \) is an upper bound for \( A \), and hence \( a \leq \sup(A) \). Adding \( c \) to both sides of this inequality yields \( c + a \leq c + \sup(A) \). Since this is true for all \( a \in A \), we have that \( c + \sup(A) \) is an upper bound for \( c + A \). By Definition 3 axiom (S2), it follows that \( \sup(c + A) \leq c + \sup(A) \).

- \( (\geq) \). Definition 3 axiom (S1) implies that \( \sup(c + A) \) is a upper bound for \( c + A \), and therefore given any element \( a \in A \), \( c + a \leq \sup(c + A) \). Subtracting \( c \) from both sides yields \( a \leq \sup(c + A) - c \). Since this is true for all \( a \in A \), we have that \( \sup(c + A) - c \) is an upper bound for \( A \). By Definition 3 axiom (S2), it follows that \( \sup(A) \leq \sup(c + A) - c \), and therefore adding \( c \) to both sides yields \( c + \sup(A) \leq \sup(c + A) \).

(b) Homework.

(c) Homework.

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### Theorem 2: Least Upper Bound Property for \( \mathbb{Q} \)

The ordered field of rational numbers \((\mathbb{Q}, +, \times, <)\) does **not** have the least upper bound property.

Before we prove the theorem, we first prove a lemma.

**Lemma 1: Irrationality of \( \sqrt{2} \)**

There is no rational number whose square is 2. (Sometimes this is stated as “\( \sqrt{2} \) is irrational”).

**Proof of Lemma.** Suppose there exists some \( p/q \in \mathbb{Q} \) such that \( (p/q)^2 = 2 \). By ??, we can assume that the fraction \( p/q \) is in lowest terms, meaning that \( p \) and \( q \) have no common prime factors. Compute

\[
\left( \frac{p}{q} \right)^2 = 2 \\
\Rightarrow \frac{p^2}{q^2} = 2 \\
\Rightarrow (1) p^2 = 2q^2 \text{ in } \mathbb{Z}.
\]

Here (1) follows from the definition of equality of two rational numbers. Therefore \( p^2 \) is an even integer. Hence \( p \) must be even as well, which means there is some \( r \in \mathbb{Z} \) such that \( 2r = p \). We now substitute \( 2r \) in for \( p \), and compute
\[ p^2 = 2q^2 \]
\[ \implies (2r)^2 = 2q^2 \]
\[ \implies 4r^2 = 2q^2 \]
\[ \implies 2r^2 = q^2. \]

Therefore \( q^2 \) is also even, so \( q \) is even too. So both \( p \) and \( q \) are even, meaning they are both divisible by the prime number 2. However we chose \( p \) and \( q \) to not have any common prime factor. This is a contradiction, which implies that our assumption was incorrect. Therefore there does not exist a rational number \( p/q \in \mathbb{Q} \) whose square is 2.

**Note 4: How to Think about \( \sqrt{2} \)**

One way to interpret this lemma is that there is something “missing” from \( \mathbb{Q} \). In fact \( \sqrt{2} \) is not the only thing “missing”, in this sense, as we will see.

**Exercise 6: Irrationality of Other Roots (Abbott Exercise 1.2.1)**

(a) Prove that \( \sqrt{3} \) is irrational. Does a similar argument work to show that \( \sqrt{6} \) is irrational?

(b) Where does the proof of Lemma 1 break down if we try to use it to prove \( \sqrt{4} \) is irrational?

**Proof.** Homework.

**Proof of Theorem.** Define

\[ A := \{ p/q \in \mathbb{Q} \mid (p/q)^2 \leq 2 \}. \]

- First we show that this set is bounded above by 2. Suppose, for contradiction, that \( p/q \in A \), but that \( p/q > 2 \). By ??, we have that \((p/q)^2 > 2^2 = 4\), which contradicts the assumption that \( p/q \in A \). Therefore indeed 2 is an upper bound for \( A \).

- However we claim that \( A \subseteq \mathbb{Q} \) does not have a supremum. Suppose, for contradiction, that \( x \in \mathbb{Q} \) is a supremum for \( A \). Let

\[ y = x - \frac{x^2 - 2}{x + 2} = \frac{2(x + 1)}{x + 2}, \]

and note that we can write

\[ y^2 - 2 = \frac{4(x + 1)^2}{(x + 2)^2} - \frac{2(x + 2)^2}{(x + 2)^2} = \frac{2(x^2 - 2)}{(x + 2)^2}. \]

By ?? axiom (O1), exactly one of the following is true: \( x^2 < 2, x^2 = 2, x^2 > 2 \). We examine each case separately.

- \( (x^2 < 2) \). If \( x^2 < 2 \), then \( x^2 - 2 < 0 \), hence Equation 1 implies that \( y > x \). On the other hand, Equation 2 implies that \( y^2 - 2 < 0 \), and therefore that \( y^2 < 2 \). Hence there exists a rational number \( y \) which satisfies \( y^2 < 2 \), and therefore \( y \in A \), yet, \( x < y \). This contradicts that \( x \) is an upper bound for \( A \), which is Definition 3 axiom (S1).

- \( (x^2 = 2) \). We proved in Lemma 1 that no rational number \( y \) can satisfy \( y^2 = 2 \), hence this case cannot occur.

- \( (x^2 > 2) \). If \( x^2 > 2 \), then \( x^2 - 2 > 0 \), and Equation 1 implies that \( y < x \), and Equation 2 implies that \( y^2 - 2 > 0 \), and therefore \( y^2 > 2 \). Hence \( y \) is an upper bound for \( A \), and yet \( y < x \), which contradicts Definition 3 axiom (S2).

\[ \Box \]