These solutions are intended to indicate roughly how much you would be expected to write. Comments in [square brackets] are additional and would not be required.

1 (32 pts.) Determine whether or not the following limits exist, and calculate them. If the limit does not exist as a number, state whether or not it can be written as $\infty$ or $-\infty$.

(a) $\lim_{x \to 3} \frac{x-3}{x^2-2x-3}$

Answer:

$$\lim_{x \to 3} \frac{x-3}{x^2-2x-3} = \lim_{x \to 3} \frac{x-3}{(x-3)(x+1)} = \lim_{x \to 3} \frac{1}{x+1} = \frac{1}{4}.$$ 

(b) $\lim_{t \to 0} \frac{\sqrt{t+1}-1}{t}$

Answer:

$$\lim_{t \to 0} \frac{\sqrt{t+1}-1}{t} = \lim_{t \to 0} \frac{(\sqrt{t+1}-1)(\sqrt{t+1}+1)}{t(\sqrt{t+1}+1)}$$
$$= \lim_{t \to 0} \frac{t+1-1}{t(\sqrt{t+1}+1)}$$
$$= \lim_{t \to 0} \frac{1}{\sqrt{t+1}+1} = \frac{1}{2}.$$ 

(c) $\lim_{x \to 1^+} \frac{2-x^2}{1-x}$

Answer: $\lim_{x \to 1^+} \frac{2-x^2}{1-x} = \lim_{x \to 1^+} (2-x^2) \cdot \frac{1}{1-x}$. Now, $\lim_{x \to 1^+} (2-x^2) = 1$. As $x$ goes to 1 from above, $1-x$ is negative, and approaches 0 from below. So $\lim_{x \to 1^+} \frac{1}{1-x} = -\infty$. Multiplying these, $\lim_{x \to 1^+} \frac{2-x^2}{1-x} = -\infty$.

(d) $\lim_{x \to \infty} \sin x$

Answer: $\lim_{x \to \infty} \sin x$ does not exist, and cannot be written as $\infty$ or $-\infty$. This is because no matter how big you choose $M$, one can always find $x, y > M$ such that $\sin x = 1$ and $\sin y = -1$. 

2 (8 pts.) Show that \( \lim_{x \to 0} x \sin(1/x) = 0 \).

**Answer:** There are two ways to do this, but both start with the observation that \(-1 \leq \sin(1/x) \leq 1\) for any \(x \neq 0\). We multiply by \(x\) on all sides: for \(x > 0\), we get \(-x \leq x \sin(1/x) \leq x\), and for \(x < 0\) we get \(x \leq x \sin(1/x) \leq -x\). We can handle both cases at once by writing \(-|x| \leq x \sin(1/x) \leq |x|\). Now, we know that \(\lim_{x \to 0} |x| = \lim_{x \to 0} (-|x|) = 0\), so the first approach is to conclude \(\lim_{x \to 0} x \sin(1/x) = 0\) from the sandwich theorem. Alternatively, we can use the definition of limit: given \(\epsilon > 0\), we want \(\delta > 0\) so that if \(0 < |x| < \delta\), then \(|x \sin(1/x)| < \epsilon\). But the above inequalities tell us that \(|x \sin(1/x)| < |x|\), so we can just choose \(\delta = \epsilon\).

3 (12 pts.) In terms of \(\epsilon\) and \(\delta\), show that \(\lim_{x \to -3} x^2 = 9\).

**Answer:** Given \(\epsilon > 0\), start with \(|x^2 - 9| < \epsilon\). [We want to replace this with inequalities in terms of \(x - (-3)\).] We do:

\[-\epsilon < x^2 - 9 < \epsilon\]
\[9 - \epsilon < x^2 < 9 + \epsilon\]
\[-\sqrt{9 - \epsilon} > x > -\sqrt{9 + \epsilon} \quad \text{(if } \epsilon \leq 9)\]
\[3 - \sqrt{9 - \epsilon} > x + 3 > 3 - \sqrt{9 + \epsilon}\]

[Note the trick here that because we are interested in \(x\) near \(-3\), we had to take negative square roots on both sides to get a useful answer] So when \(\epsilon \leq 9\), we can choose \(\delta\) to be the smaller of \(3 - \sqrt{9 - \epsilon}\) and \(\sqrt{9 + \epsilon} - 3\). For \(\epsilon > 9\), \(\delta = \min\{3, \sqrt{18} - 3\}\) works.
4 (20 pts.) For each question, answer only “true” or “false”. You will receive 5 points for a correct answer, 2 points for no answer, and 0 points for an incorrect answer. There is no partial credit.

(a) If \( f(x) > g(x) \) on \((0, 1)\), and both \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^+} g(x) \) exist, then
\[
\lim_{x \to 0^+} f(x) > \lim_{x \to 0^+} g(x).
\]

Answer: False. [For instance, consider \( f(x) = 2x \) and \( g(x) = x \).]

(b) If \( \lim_{x \to 0^+} f(x) \) is a positive number, then there is some interval \((0, c)\) for \( c > 0 \) on which \( f(x) \) is positive.

Answer: True. [If the limit is \( L \), choose \( \epsilon \) less than \( L \), and set \( c = \delta \).]

(c) The function \( f(x) = |x|/x \) has a removable discontinuity at \( x = 0 \).

Answer: False. [The limit doesn’t exist]

(d) For every function \( f(x) \) such that \( \lim_{x \to 0} f(x) = 0 \) and every \( g(x) \) such that \( \lim_{x \to 0} g(x) = \infty \), the limit \( \lim_{x \to 0} f(x) \cdot g(x) \) does not exist.

Answer: False. [For instance, consider \( f(x) = x^2 \) and \( g(x) = 1/x \).]
Consider the function
\[ f(x) = \begin{cases} 
\frac{x^3 - 2x^2 + 2x - 4}{x^2 - 3x + 2} : & x \neq 1, 2 \\
4 : & x = 1, 2 
\end{cases} \]

At which points is this function continuous, and at which points is it discontinuous? For each discontinuity, say whether or not it is removable. (Hint: factor the denominator first) Find all asymptotes to the graph. Sketch the graph.

**Answer:** The denominator is 0 only at \( x = 1, 2 \), so \( f(x) \) is continuous away from these points. To analyze \( x = 1, 2 \), we first factor. The denominator factors as \((x - 1)(x - 2)\). Plugging in \( x = 1, 2 \), we see the numerator is 0 at \( x = 2 \), so we can factor it \((x - 2)(x^2 + 2)\). Away from \( x = 2 \), we can cancel and \( f(x) \) is the same as \( g(x) = \frac{x^2 + 2}{x-1} \). \( g(x) \) is continuous at \( x = 2 \), so we see that

\[ \lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = g(2) = 6. \]

But \( f(2) = 4 \), so \( f(x) \) is not continuous at \( x = 2 \). This is a removable discontinuity, since the limit does exist. On the other hand,

\[ \lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} (x^2 + 2) \cdot \frac{1}{x - 1}. \]

Now, \( \lim_{x \to 1} (x^2 + 2) = 3 \), and \( \lim_{x \to 1} \frac{1}{x - 1} = \infty \), so \( \lim_{x \to 1} f(x) = \infty \).

But \( \lim_{x \to 1^-} \frac{1}{x - 1} = -\infty \), so \( \lim_{x \to 1^-} f(x) = -\infty \), and we conclude that \( \lim_{x \to 1} f(x) \) does not exist. Thus, \( f(x) \) is discontinuous at \( x = 1 \), and the discontinuity is not removable.

The above calculation shows that there is a vertical asymptote at \( x = 1 \), and this is the only vertical asymptote. For horizontal and oblique asymptotes, look at \( f(x) \) as \( x \to \pm \infty \). Using \( f(x) = g(x) \) for \( x \neq 2 \), look instead at \( \frac{x^2 + 2}{x-1} = x + 1 + \frac{3}{x-1} \). Now, \( \lim_{x \to \infty} \frac{3}{x-1} = 0 \), and also \( \lim_{x \to -\infty} \frac{3}{x-1} = 0 \), so the line \( y = x + 1 \) is an oblique asymptote for \( f(x) \) in both directions.

[Graph omitted; I can sketch it in class on Wednesday.]
6 (12 pts.) For the function $f(x) = \sqrt{x}$, what is the average rate of change of $f(x)$ from $x = 3$ to $x = 4$? What is the instantaneous rate of change at $x = 3$? (Compute this from our definitions, without using derivative laws)

**Answer:** The average rate of change is

\[ \frac{\sqrt{4} - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}. \]

The instantaneous rate of change is

\[
\lim_{h \to 0} \frac{\sqrt{3 + h} - \sqrt{3}}{h} = \lim_{h \to 0} \frac{(\sqrt{3 + h} - \sqrt{3})(\sqrt{3 + h} + \sqrt{3})}{h(\sqrt{3 + h} + \sqrt{3})} \\
= \lim_{h \to 0} \frac{3 + h - 3}{h(\sqrt{3 + h} + \sqrt{3})} \\
= \lim_{h \to 0} \frac{1}{\sqrt{3 + h} + \sqrt{3}} = \frac{1}{2\sqrt{3}}.
\]