RULES FOR INFINITE LIMITS

BRIAN OSSERMAN

The book does not give any rules for working with limits equal to \( \pm \infty \). One can use some of the usual limit rules in this context, but it is important to be careful: if \( \lim_{x \to c} f(x) = \infty \) and \( \lim_{x \to c} g(x) = \infty \), then \( \lim_{x \to c}(f(x) + g(x)) = \infty \), but \( \lim_{x \to c}(f(x) - g(x)) \) could be \( \infty \), a finite number, \(-\infty\), or completely undefined.

We can summarize how the limit laws are adapted to infinite limits as follows. If \( L \) is any number (but not infinite), then:

- \( \infty + L = \infty \)
- \( -\infty + L = -\infty \)
- \( \infty + \infty = \infty \)
- \( -\infty - \infty = -\infty \)
- \( \infty - \infty = ?!?! \)
- \( \infty \cdot L = \infty \) if \( L > 0 \)
- \( \infty \cdot L = -\infty \) if \( L < 0 \)
- \( \infty \cdot \infty = \infty \)
- \( \infty \cdot -\infty = -\infty \)
- \( \infty \cdot 0 = ?!?! \)
- \((\infty)^n = \infty \) for \( n > 0 \)
- \((-\infty)^n = (-1)^n \infty \) for \( n > 0 \)
- \( \sqrt[n]{\infty} = \infty \) for \( n > 0 \)
- \( \sqrt[n]{-\infty} = -\infty \) for \( n > 0 \) and odd
- \( \frac{L}{\infty} = 0 \).

The two marked ?!?! mean that we don’t get any information, and have to work some more with the particular functions to figure out what is going on. The limit might be a number, or \( \pm \infty \), or there might not be any limit (even \( \pm \infty \)). I’m including them to remind you that if you see those two situations, you cannot try to apply a rule directly, and should instead try to simplify further.

One more rule which is a bit harder to state:

“I \( \frac{L}{0^+} = \infty \)” and “I \( \frac{L}{0^-} = -\infty \), for \( L > 0 \).

Here, \( 0^+ \) means that the limit in the denominator is zero, but the function is positive in the relevant interval (for instance, if we are looking at a limit as \( x \to c \), the function has to be positive for \( x \) near \( c \) on both sides; if we are looking at a limit as \( x \to c^+ \), we only need to look at \( x \) near \( c \) and also greater than \( c \).

More precisely, for two-sided limits the rule \( \frac{L}{0^+} = \infty \) means that \( \lim_{x \to c} \frac{f(x)}{g(x)} = \infty \) if:

- \( \lim_{x \to c} f(x) = L > 0 \),
- \( \lim_{x \to c} g(x) = 0 \), and
- \( g(x) > 0 \) for every \( x \) sufficiently near \( c \) (except possibly \( x = c \)).
Below are two examples.

**Example.** \( \lim_{x \to 5^-} \frac{1}{x-5} \).
\[ \lim_{x \to 5^-} x - 5 = 0, \text{ and for } x < 5, \text{ the function } x - 5 \text{ is negative, so we can think of it as } \lim_{x \to 5^-} x - 5 = 0^- \text{. Then by the above rule (since } 1 > 0), \text{ we get} \]
\[ \lim_{x \to 5^-} \frac{1}{x-5} = \frac{1}{0^-} = -\infty. \]

**Example.** \( \lim_{x \to 2^+} \frac{x+1}{x-2} \). A couple of different approaches:
\[ \lim_{x \to 2^+} x + 1 = 3. \lim_{x \to 2^+} \frac{1}{x-2} = 1/0^+ = \infty, \text{ as in the above example. Then} \]
\[ \lim_{x \to 2^+} \frac{x+1}{x-2} = 3 \cdot \infty = \infty. \]

Or, break it into a sum:
\[ \lim_{x \to 2^+} \frac{x+1}{x-2} = \lim_{x \to 2^+} \left( 1 + \frac{3}{x-2} \right) = 1 + \frac{3}{0^+} = 1 + \infty = \infty. \]