The point of this note is to help you try to understand the formal definition of a limit, and how to use it.

About the definition. Let’s start by recalling the definition.

**Definition.** \( \lim_{x \to c} f(x) = L \) if, for every number \( \epsilon > 0 \), there is some number \( \delta > 0 \) with the property that for every number \( x \), if \( 0 < |x - c| < \delta \) then \( |f(x) - L| < \epsilon \).

I emphasize again that \( \epsilon \) (epsilon) and \( \delta \) (delta) simply stand for numbers, just like \( c \) does.

We will break this definition down to understand it better. We know that \( |f(x) - L| < \epsilon \) is the same as saying that the distance between \( f(x) \) and \( L \) is less than \( \epsilon \), or equivalently, that \( f(x) \) is within \(^1\epsilon \) of \( L \). We can also write this as “\( f(x) \) is in the interval \((L - \epsilon, L + \epsilon)\).” Another way we’ve expressed it is by the pair of inequalities \(-\epsilon < f(x) - L < \epsilon\).

Similarly, \( 0 < |x - c| < \delta \) just says that \( x \) is within \( \delta \) of \( c \), but not equal to \( c \). We can write it as “\( x - c \) is in the union of intervals \((c - \delta, c) \cup (c, c + \delta)\).” It’s also the same as saying that \( x \neq c \) and \(-\delta < x - c < \delta\).

So, suppose \( \epsilon \) and \( \delta \) are fixed. What does it mean to say that for every number \( x \), if \( 0 < |x - c| < \delta \), then \( |f(x) - L| < \epsilon \)? It says that for every \( x \) which is within \( \delta \) of \( c \) (except possibly \( x = c \) itself), we have that \( f(x) \) is within \( \epsilon \) of \( L \). Phrasing it slightly differently, it says that if we want to guarantee that \( f(x) \) is within \( \epsilon \) of \( L \), we can achieve this by restricting \( x \) to be within \( \delta \) of \( c \) (again, except possibly at \( x = c \)).

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\(^{1}\)for convenience, when I say “within” in this note, I mean strict inequality, not less than or equal to.
Visually, it says that every point of the graph between the lines \(x = c - \delta\) and \(x = c + \delta\) (again, except possibly the point on \(x = c\) itself) must lie between the lines \(y = L - \epsilon\) and \(y = L + \epsilon\).

So, finally, what does the definition of limit say? It says that \(f(x)\) gets closer and closer to \(L\) as \(x\) gets closer and closer to \(c\). More precisely, it says that no matter how small an \(\epsilon\) you give me (as long as it’s positive), if you want to guarantee that \(f(x)\) is within \(\epsilon\) of \(L\), I can find some (possibly tiny) interval \((c - \delta, c + \delta)\) around \(x = c\) on which you get what you want – that is, for every \(x\) in the interval (except possibly \(x = c\)) we have that \(f(x)\) is guaranteed to be within \(\epsilon\) of \(L\).

You can think of this in terms of error control: you want to make sure that the value of \(f(x)\) can’t be off by more than some \(\epsilon\) from a target value \(L\), and you want to know how close \(x\) has to be to \(c\) in order to guarantee this. This may tell you how precise a measurement of \(x\) has to be, or what kind of error tolerance to allow in manufacturing.

Hopefully, with perseverance you will understand the conceptual idea of a limit, and the formal definition, and you will also understand why the formal definition corresponds to our intuitive concept!

**Working examples.** We can summarize the procedure for calculating \(\delta\) from \(\epsilon\) as follows:

- Start with the inequality \(|f(x) - L| < \epsilon\), and replace it by the equivalent pair of inequalities \(-\epsilon < f(x) - L < \epsilon\).
- Use algebraic manipulations to derive inequalities in terms of \(x - c\), usually of the form \(-t_1 < x - c < t_2\), where \(t_1\) and \(t_2\) are positive numbers (or expressions in terms of \(\epsilon\) which are always positive when \(\epsilon\) is small enough).
- Set \(\delta\) to be the smaller of \(t_1\) and \(t_2\).

Because the idea is that we are interested in working with small values of \(\epsilon\), you are always allowed to restrict to \(\epsilon\) less than a given (positive) number during your algebraic manipulations, but you should make this restriction clear.

Let’s return to the most complicated example from class and look at it further.

**Example.** Show from the definition that \(\lim_{x \to 1} x^2 + 2 = 3\).

**Solution:** Given \(\epsilon > 0\), the following lines are equivalent:

\[
| (x^2 + 2) - 3 | < \epsilon \\
-\epsilon < (x^2 + 2) - 3 < \epsilon \\
-\epsilon < x^2 - 1 < \epsilon \\
-\epsilon + 1 < x^2 < \epsilon + 1 \\
\sqrt{\epsilon + 1} - 1 < x < \sqrt{\epsilon + 1} \quad \text{(assuming } \epsilon \leq 1 \text{ and } x > 0) \\
\sqrt{-\epsilon + 1} - 1 < x - 1 < \sqrt{\epsilon + 1} - 1
\]

So, if \(|x - 1| < \sqrt{\epsilon + 1} - 1\) and \(|x - 1| < 1 - \sqrt{1 - \epsilon}\), this guarantees \(|(x^2 + 2) - 3| < \epsilon\). Now, both \(\sqrt{\epsilon + 1} - 1\) and \(1 - \sqrt{1 - \epsilon}\) are positive, so we simply set \(\delta\) to be the minimum of these two numbers.

What is shown above is all that is needed for working a problem of this type. If we actually want to compute \(\delta\), we have to figure out which number is smaller. This is easy for any particular value of \(\epsilon\), but takes more work to solve for all \(\epsilon\)s at once.
Because a number of students asked about it, let’s discuss the restriction $\epsilon \leq 1$, and why it is okay. First off, the above calculation says that if $\epsilon = 1$, then we can choose $\delta = \min\{\sqrt{2} - 1, 1 - \sqrt{0}\} = \sqrt{2} - 1$. Now, we know that if a $\delta$ works for a particular $\epsilon$, it also works for any larger $\epsilon$, so if $\epsilon > 1$, we can still set $\delta = \sqrt{2} - 1$, and this will be fine. That is, for all $\epsilon$, even $\epsilon > 1$, we have some explicit choice of $\delta$ which works.

However, this might not be the best (i.e., largest) value of $\delta$, so let’s just revisit the calculation assuming $\epsilon > 1$. In this case, we ran into trouble when we wanted to take the square root of the inequalities $-\epsilon + 1 < x^2 < \epsilon + 1$.

But if $\epsilon > 1$, the left-hand side is negative, while $x^2$ is always at least 0. This means that if $\epsilon > 1$, the left inequality is always satisfied, and we can throw it away! (You can also see this in terms of the first version of the inequality: if $\epsilon > 1$, then $-\epsilon < (x^2 + 2) - 3$ always, because $x^2 + 2 - 3 \geq -1$) That means that to get what we want, for $\epsilon > 1$ we can just work with the right-hand inequality, and then we can just set $\delta = \sqrt{\epsilon + 1} - 1$. So for instance, if $\epsilon = 2$, we can set $\delta = \sqrt{3} - 1$, which is better than what we had before.

The other source of confusion involves the choice of positive square root. The reasoning is that since we are taking a limit as $x \to 1$, we are interested in positive values of $x$, and since we were getting $x$ in the middle of the inequalities after taking square roots, we’d better use positive square roots. To illustrate this, let’s take the same example, with a negative value of $c$.

Example. Show from the definition that $\lim_{x \to -2} x^2 + 2 = 6$.

Solution: Given $\epsilon > 0$, the following lines are equivalent:

$|x^2 + 2 - 6| < \epsilon$

$-\epsilon < x^2 - 4 < \epsilon$

$-\epsilon + 4 < x^2 < \epsilon + 4$

$-\sqrt{-\epsilon + 4} < x < \sqrt{\epsilon + 4}$ (assuming $\epsilon \leq 4$ and $x < 0$)

So, if $|x - (-2)| < -\sqrt{-\epsilon + 4} - (-2) = 2 - \sqrt{-\epsilon + 4}$ and $|x - (-2)| < \sqrt{\epsilon + 4} - 2$, this guarantees $|(x^2 + 2) - 6| < \epsilon$. Now, both $2 - \sqrt{-\epsilon + 4}$ and $\sqrt{\epsilon + 4} - 2$ are positive, so we simply set $\delta$ to be the minimum of these two numbers.

Notice that when we took the negative square root, we also had to reverse the inequalities. Also, we still used $x$ in the middle, instead of $-x$, because now $x$ itself is supposed to be negative. If you want, you could instead take the positive square roots on the left and right, but put $-x$ in the middle. This would ultimately give the same answer.

Using the definition for proofs. To wrap up, for the more theory-inclined among you, since there’s no time in lecture to illustrate using the formal definition to prove theorems, I will give a few quick examples here.

Proof of “Identity rule”. We want to show that $\lim_{x \to c} x = c$ for any $c$. Given $\epsilon > 0$, we want $\delta > 0$ with the property that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. We can just choose $\delta = \epsilon$!
Proof of “Constant rule”. We want to show that \( \lim_{x \to c} k = k \) for any \( c \) and \( k \). Given \( \epsilon > 0 \), we want \( \delta > 0 \) with the property that if \( 0 < |x - c| < \delta \), then \( | \frac{f(x) - L}{k - k} | < \epsilon \). But \( |k - k| = 0 \), so any \( \delta \) works!

Proof of Sum rule. Suppose \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \); we want to show \( \lim_{x \to c} (f(x) + g(x)) = L + M \). Given \( \epsilon > 0 \), we want \( \delta > 0 \) so that if \( 0 < |x - c| < \delta \), then \( |f(x) + g(x) - (L + M)| < \epsilon \). This requires a slight trick. Using the limits of \( f(x) \) and \( g(x) \), and replacing \( \epsilon \) by \( \epsilon/2 \), we know there exists \( \delta_1 \) such that if \( 0 < |x - c| < \delta_1 \), then \( |f(x) - L| < \epsilon/2 \), and \( \delta_2 \) such that if \( 0 < |x - c| < \delta_2 \), then \( |g(x) - M| < \epsilon/2 \). Set \( \delta \) to be the smaller of \( \delta_1 \) and \( \delta_2 \), so that if \( 0 < |x - c| < \delta \), then \( 0 < |x - c| < \delta_1 \) and \( 0 < |x - c| < \delta_2 \). Let’s check that this \( \delta \) works.

If \( 0 < |x - c| < \delta \), then
\[
|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)| \\
\leq |f(x) - L| + |g(x) - M| \\
< \epsilon/2 + \epsilon/2 = \epsilon.
\]

This proves the sum rule.

Proof of bounding theorem. Finally, we prove the theorem that if \( f(x) \leq g(x) \) near \( x = c \), and \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \), then \( L \leq M \). This proof is by contradiction: if \( L > M \), we can choose \( \epsilon < (L - M)/2 \). Then there is some \( \delta_1 \) so that if \( 0 < |x - c| < \delta_1 \), then \( |f(x) - L| < \epsilon \), and some \( \delta_2 \) so that if \( 0 < |x - c| < \delta_2 \), then \( |g(x) - M| < \epsilon \). Now choose some \( x \) with \( 0 < |x - c| < \delta_1 \) and \( 0 < |x - c| < \delta_2 \). Then \( f(x) - L > -\epsilon \) and \( g(x) - M < \epsilon \), so \( M - g(x) > -\epsilon \). Adding these, we find
\[
f(x) - L + M - g(x) > -2\epsilon > M - L.
\]

Adding \( L - M \) to both sides, we find \( f(x) - g(x) > 0 \), which is a contradiction. We conclude that we could not have had \( L > M \) in the first place.