Recall that if $f$ is continuous on $[a, b]$, then we know that $f$ has both an absolute maximum and an absolute minimum, and we have the following procedure to find them:

1. Find all the critical points of $f$.
2. Evaluate $f$ at all critical points and endpoints.
3. Take the largest and smallest values to find the absolute maximum value and absolute minimum value of $f$, respectively. Take the points at which these values occurred to find where $f$ has its absolute maxima and minima.

What about if $f$ is continuous, but its domain is not a closed, finite interval? If $f$ has absolute maxima, then the procedure will work to find it, and similarly for absolute minima. However, if $f$ doesn’t have an absolute maximum or minimum, then the procedure is almost guaranteed to give the wrong answer. So, in these cases, we need to modify our procedure.

Often, we can use limits for this, as follows: if $f$ is defined on $(a, b)$, and the limit of $f(x)$ as $x \to a$ and $x \to b$ are defined, then the absolute maximum exists if and only if there is some value on our list of critical points which has value at least as big as the two limits. The absolute minimum exists if and only if some value on our list is at least as small as the limits. If either limit is equal to $\infty$, the absolute maximum does not exist, but the absolute minimum is not affected. If either limit is equal to $-\infty$, the absolute minimum does not exist, but the absolute maximum is not affected.

This works if $a$ and/or $b$ are infinite. The same principle also applies to half-open intervals: we take limits at open ends, and evaluate endpoints at closed ends.

Thus, the modified procedure is that we should go through our original procedure, evaluating at critical points and closed endpoints. Then we should take the limits at open endpoints and use them to determine whether or not each of the absolute maximum and absolute minimum exist.

**Example.** Find the absolute maxima and minima of $f(x) = \sqrt{x}$.

**Solution:** The domain of $f$ is $[0, \infty)$. Although $f$ is continuous, since this is not a closed, finite interval, we have to decide whether or not it has absolute maxima and minima. First, $f'(x) = \frac{1}{2\sqrt{x}}$, which is defined (and non-zero) for $x > 0$. Since $x = 0$ is an endpoint, we see that $f$ has no critical points. Its only endpoint is $x = 0$, where the value is 0.

Now, since the domain is open at the right endpoint, we consider the limit $\lim_{x \to \infty} \sqrt{x}$. This is equal to $\infty$, so $f$ has no absolute maximum. On the other hand, $f$ does have an absolute minimum, which must occur at the endpoint $x = 0$.

**Example.** Find the absolute extrema of $f(x) = |x^2 - 4|$, on the interval $(-3, 2.5]$. 
Solution: First, \( f(x) \) is continuous. The only endpoint is \( x = 2.5 \). Next, to find the critical points, since \( x^2 - 4 \geq 0 \) exactly when \( |x| \geq 2 \), we can write

\[
 f(x) = \begin{cases} 
 x^2 - 4 & \text{if } x \text{ is in } (-3, -2] \text{ or } [2, 2.5] \\
 4 - x^2 & \text{if } x \text{ is in } (-2, 2). 
\end{cases}
\]

Then we get the \( f'(x) = 2x \) for \( x \) in \((-3, -2)\) or \((2, 2.5)\) and \( f'(x) = -2x \) for \( x \) in \((-2, 2)\). At \( x = -2 \), these two derivative formulas are different, so \( f \) doesn’t have a derivative, and the same is true for \( x = 2 \), so these are both critical points. We see that \( f'(x) = 0 \) only for \( x = 0 \), so this is also a critical point, and the absolute extrema must occur either at \(-2, 0\) or 2 or at the endpoint 2.5 if they exist. We evaluate \( f \) at these points:

\[
 f(-2) = 0, \quad f(0) = 4, \quad f(2) = 0, \quad f(2.5) = 2.25.
\]

So, if the absolute maximum exists, its value is 4 and occurs at \( x = 0 \), and if the absolute minimum exists, its value is 0 and occurs at \( x = \pm 2 \).

To determine whether or not they exist, we also consider the limit at the open end of the interval:

\[
 \lim_{x \to -3} |x^2 - 4| = |(-3)^2 - 4| = 5.
\]

Since this is bigger than 4, it means that in fact \( f \) does not have an absolute maximum on \((-3, 2.5)\). On the other hand, since it is not smaller than 0, we see that \( f \) does have the absolute minimum at \( x = \pm 2 \), with value 0.

Sometimes, we do not know how to take a limit (or it doesn’t exist), so we are forced to use other methods to determine whether or not an absolute maximum or minimum occurs.

Example. For the function \( f(x) = (x^2 - 6x + 9)e^x \), find the critical points, the intervals where \( f \) is increasing and decreasing, and the local and absolute extrema.

Solution: \( f \) is continuous and differentiable on its entire domain, which is \((-\infty, \infty)\). The derivative is

\[
 f'(x) = (x^2 - 6x + 9)e^x + (2x - 6)e^x = (x^2 - 4x + 3)e^x.
\]

Since \( e^x \) is never 0, we see that \( f'(x) = 0 \) precisely when \( x^2 - 4x + 3 = 0 \). Factoring this as \((x - 1)(x - 3)\), we get that the critical points occur at \( x = 1 \) and \( x = 3 \). There are no endpoints to consider, so we just evaluate \( f \) at the critical points: \( f(1) = 4e \), and \( f(3) = 0 \).

To determine if these are absolute extrema, we have to evaluate limits as \( x \to \infty \) and \( x \to -\infty \). We have

\[
 \lim_{x \to \infty} (x^2 - 6x + 9)e^x = \infty \cdot \infty = \infty,
\]

so \( f \) has no absolute maximum. On the other hand,

\[
 \lim_{x \to -\infty} (x^2 - 6x + 9)e^x = \infty \cdot 0 = ?!?!.
\]

Actually, this limit is 0, but we have not yet learned how to evaluate it. However, we can still determine the absolute minimum, by observing that \( x^2 - 6x + 9 = (x - 3)^2 \) is always nonnegative, so \( f(x) \) is also always nonnegative, and the critical point \( x = 3 \), where \( f \) is equal to 0, must be the absolute minimum.