1. Definitions

Prologue.

Definition 1.1. A secant line is the line connecting two points on a graph. A tangent line is the line through a single point on the graph, with the same slope as the graph at that point (slope to be defined later).

Definition 1.2. The average rate of change of \( f(x) \) between \( x_0 \) and \( x_0 + h \) is

\[
\frac{f(x_0 + h) - f(x_0)}{h}.
\]

The instantaneous rate of change of \( f(x) \) at \( x_0 \) obtained from the above by letting \( h \) approach 0.

Limits.

Definition 1.3. We say the limit of \( f(x) \) as \( x \) approaches \( c \) is a number \( L \), and write

\[
\lim_{x \to c} f(x) = L,
\]

if for every number \( \epsilon > 0 \), there is some number \( \delta > 0 \) with the property that for every number \( x \), if \( 0 < |x - c| < \delta \) then \( |f(x) - L| < \epsilon \).

We say the limit of \( f(x) \) as \( x \) approaches \( c \) exists if there is some number \( L \) such that \( \lim_{x \to c} f(x) = L \).

Informally, \( \lim_{x \to c} f(x) = L \) if \( f(x) \) is arbitrarily close to \( L \) as \( x \) gets closer and closer to \( c \).

Definition 1.4. \( f(x) \) has right-hand limit \( L \) at \( c \), written \( \lim_{x \to c^+} f(x) = L \), if for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( c < x < c + \delta \), then \( |f(x) - L| < \epsilon \).

Similarly, \( f(x) \) has left-hand limit \( L \) at \( c \), written \( \lim_{x \to c^-} f(x) = L \), if for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( c - \delta < x < c \), then \( |f(x) - L| < \epsilon \).

Definition 1.5. \( f(x) \) has limit \( L \) as \( x \) approaches \( \infty \), written \( \lim_{x \to \infty} f(x) = L \), if for all \( \epsilon > 0 \), there is some \( M \) such that for every \( x > M \), we have \( |f(x) - L| < \epsilon \).

\( f(x) \) has limit \( L \) as \( x \) approaches \( -\infty \), written \( \lim_{x \to -\infty} f(x) = L \), if for all \( \epsilon > 0 \), there is some \( M \) such that for every \( x < M \), we have \( |f(x) - L| < \epsilon \).

Definition 1.6. \( \lim_{x \to c} f(x) = \infty \) if for all numbers \( B \), there is some \( \delta > 0 \) such that for every \( x \) with \( 0 < |x - c| < \delta \), we have \( f(x) > B \).

\( \lim_{x \to c} f(x) = -\infty \) if for all numbers \( B \), there is some \( \delta > 0 \) such that for every \( x \) with \( 0 < |x - c| < \delta \), we have \( f(x) < B \).

Remember that the infinite limit definitions can be combined with one another, and/or with one-sided limits, and we’re not including all the combinations above.
Continuity.

**Definition 1.7.** A function $f(x)$ is **continuous** at an interior point $c$ of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

$f(x)$ is **continuous** at a left endpoint $c$ of its domain if

$$\lim_{x \to c^-} f(x) = f(c),$$

and is **continuous** at a right endpoint $c$ of its domain if

$$\lim_{x \to c^+} f(x) = f(c).$$

If $f(x)$ is not continuous at $c$, we say $f(x)$ is **discontinuous** at $c$, and/or that $c$ is a **discontinuity** of $f(x)$.

If the relevant limit exists, but $f(c)$ is undefined or different from the limit, we say that $f(x)$ has a **removable discontinuity** at $c$.

**Definition 1.8.** $f(x)$ is **continuous** on an interval if it is continuous at every point of the interval. $f(x)$ is **continuous** if it is continuous at every point of its domain.

**Definition 1.9.** If $f(x)$ has a limit at $c$, but $f(c)$ is undefined, we define the **continuous extension** of $f(x)$ to $c$ to be the function $F(x)$ that is the same as $f(x)$ everywhere except $c$, but is also defined at $c$ by $F(c) = \lim_{x \to c} f(x)$.

Asymptotes.

**Definition 1.10.** A (horizontal) line $y = b$ is a (horizontal) asymptote for $f(x)$ if

$$\lim_{x \to \infty} f(x) = b, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$ 

A line $y = \ell(x) = mx + b$ is an **oblique** or slanted asymptote of $f(x)$ if $\lim_{x \to \infty}(f(x) - \ell(x)) = 0$ or $\lim_{x \to -\infty}(f(x) - \ell(x)) = 0$.

The (vertical) line $x = a$ is a (vertical) asymptote if $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

2. Results

Limits.

**Theorem 2.1.** Given numbers $L, M, c, k$, and functions $f(x), g(x)$ defined near $x = c$, and if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, the following laws apply:

1. **Sum rule:** $\lim_{x \to c}(f(x) + g(x)) = L + M$.
2. **Difference rule:** $\lim_{x \to c}(f(x) - g(x)) = L - M$.
3. **Constant multiple rule:** $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$.
4. **Product rule:** $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$.
5. **Quotient rule:** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$.
6. **Power rule:** if $n$ is a positive integer, then $\lim_{x \to c}(f(x))^n = L^n$.
7. **Root rule:** if $n$ is a positive integer, and either $L > 0$ or $n$ is odd, then $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$.
8. **Identity rule:** $\lim_{x \to c} x = c$.
9. **Constant rule:** $\lim_{x \to c} k = k$. 

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Theorem 2.2. (Limits of polynomials) If

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \]

then

\[ \lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0, \]

Theorem 2.3. (Limits of rational functions) If \( P(x), Q(x) \) are polynomials, and \( Q(c) \neq 0 \), then

\[ \lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}. \]

Theorem 2.4. (Sandwich theorem) Suppose \( g(x) \leq f(x) \leq h(x) \) for all \( x \) near \( c \) (except possibly at \( x = c \)), and suppose

\[ \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L. \]

Then

\[ \lim_{x \to c} f(x) = L. \]

Theorem 2.5. If \( f(x) \leq g(x) \) for all \( x \) near \( c \) (except possibly at \( x = c \)), and if both the limit of \( f(x) \) and the limit of \( g(x) \) exist as \( x \) approaches \( c \), then

\[ \lim_{x \to c} f(x) \leq \lim_{x \to c} g(x). \]

Theorem 2.6. \( \lim_{x \to c} f(x) = L \) if and only if both \( \lim_{x \to c^+} f(x) = L \) and \( \lim_{x \to c^-} f(x) = L \).

The book doesn’t really talk about how to apply limit laws to infinite limits, but it’s good to have some tools. The basic limit laws can be extended to infinite limits, but one has to be careful: if \( \lim_{x \to c} f(x) = \infty \) and \( \lim_{x \to c} g(x) = \infty \), then \( \lim_{x \to c} (f(x) + g(x)) = \infty \), but \( \lim_{x \to c} (f(x) - g(x)) \) could be \( \infty \), a finite number, \(-\infty\), or completely undefined. We can summarize how a few limit laws are adapted to infinite limits as follows:

- \( \infty + L = \infty \)
- \( -\infty + L = -\infty \)
- \( \infty + \infty = \infty \)
- \( -\infty - \infty = -\infty \)
- \( \infty - \infty = \text{?!!?} \)
- \( \infty \cdot L = \infty \) if \( L > 0 \)
- \( \infty \cdot L = -\infty \) if \( L < 0 \)
- \( \infty \cdot \infty = \infty \)
- \( \infty \cdot -\infty = -\infty \)
- \( \infty \cdot 0 = \text{?!!?} \)
- \( (\infty)^n = \infty \) for \( n > 0 \)
- \( (-\infty)^n = (-1)^n \infty \) for \( n > 0 \)
- \( \sqrt[n]{\infty} = \infty \) for \( n > 0 \)
- \( \sqrt[n]{-\infty} = -\infty \) for \( n > 0 \) and odd.
- \( 1/\infty = 0 \)
Continuity.

Theorem 2.7. If \( f(x) \) and \( g(x) \) are continuous at \( c \), then the following functions are also continuous at \( c \):

1. \( f(x) + g(x) \);
2. \( f(x) - g(x) \);
3. \( k \cdot f(x) \), for any number \( k \);
4. \( f(x) \cdot g(x) \);
5. \( f(x)/g(x) \), if \( g(c) \neq 0 \);
6. \( f(x)^n \), for \( n \) a positive integer;
7. \( \sqrt{f(x)} \), for \( n \) a positive integer, if \( f(c) > 0 \) or \( n \) is odd.

Theorem 2.8. If \( P(x) \) is a polynomial function, then \( P(x) \) is continuous on \( (-\infty, \infty) \). If \( Q(x) \) is another polynomial, then \( P(x)/Q(x) \) is continuous for all \( c \) except those with \( Q(c) = 0 \).

Theorem 2.9. If \( f(x) \) is defined on an interval and continuous, then \( f^{-1}(x) \) is also continuous.

Theorem 2.10. If \( g(y) \) is continuous at \( b \), and \( \lim_{x \to c} f(x) = b \), then

\[
\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).
\]

In particular, if \( f(x) \) is continuous at \( c \), and \( g(y) \) is continuous at \( b = f(c) \), then \( g(f(x)) \) is continuous at \( c \).

Theorem 2.11. (Intermediate Value Theorem) If \( f(x) \) is continuous on \( [a,b] \), then it takes on every value between \( f(a) \) and \( f(b) \).

3. Examples you should know

Example 3.1.

\[
f(x) = \begin{cases} 
0 : x = 0 \\
\sin \left( \frac{1}{x} \right) : x \neq 0.
\end{cases}
\]

This function oscillates more and more wildly as \( x \) gets close to 0, so \( \lim_{x \to 0} f(x) \) doesn’t exist.

Example 3.2. \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). Important: \( x \) must be in radians for this to be true.

Example 3.3. \( \lim_{x \to 0} |x| = 0 \).

Example 3.4. \( \lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0 \).

Example 3.5. \( \lim_{x \to 0^+} \frac{1}{x} = \infty \).

\[
\lim_{x \to 0^-} \frac{1}{x} = -\infty.
\]

Example 3.6. \( |x|, \cos x, \sin x, e^x, \ln x \) are continuous.
4. Procedures

Epsilon/delta calculations. We can summarize the procedure for calculating $\delta$ from $\epsilon$ as follows:

- Start with the inequality $|f(x) - L| < \epsilon$, and replace it by the equivalent pair of inequalities $-\epsilon < f(x) - L < \epsilon$.
- Use algebraic manipulations to derive inequalities in terms of $x - c$, usually of the form $-t_1 < x - c < t_2$, where $t_1$ and $t_2$ are positive numbers (or expressions in terms of $\epsilon$ which are always positive when $\epsilon$ is small enough).
- Set $\delta$ to be the smaller of $t_1$ and $t_2$.

Because the idea is that we are interested in working with small values of $\epsilon$, you are always allowed to restrict to $\epsilon$ less than a given (positive) number during your algebraic manipulations, but you should make this restriction clear.

Sketching graphs. Given a function $f(x)$, we can start to sketch its graph by finding any discontinuities and asymptotes and using these to sketch some of the main features.