REVIEW OF MATERIAL FOR EXAM 2

This review sheet is not intended to cover every individual piece of information you need to know for the exam, especially in terms of definitions, notation and examples. The purpose is to summarize the main methods and formulas we have learned, to help you study for the exam.

1. Rules for derivatives

Definition. The derivative of a function $f$ at $x$, denoted by $f'(x)$ or $\frac{df}{dx}$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists.

Theorem 1 (Basic rules). Suppose $f(x)$ and $g(x)$ are differentiable at $x$, and $c$ is a number. Then we have the following rules:

(i) $\frac{d}{dx}(c) = 0$.

(ii) $\frac{d}{dx}(x) = 1$.

(iii) $\frac{d}{dx}(cf(x)) = c \left( \frac{d}{dx}f(x) \right)$.

(iv) $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

(v) $\frac{d}{dx}(f(x)g(x)) = \left( \frac{d}{dx}f(x) \right) g(x) + f(x) \left( \frac{d}{dx}g(x) \right)$.

(vi) $\frac{d}{dx}\left( \frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx}f(x) g(x) - f(x) \frac{d}{dx}g(x)}{(g(x))^2}$, provided that $g(x) \neq 0$.

Theorem 2. (Chain rule) If $f(u)$ is differentiable at $u = g(x)$, and $g$ is differentiable at $x$, then

$$(f \circ g)'(x) = f'(g(x)) \circ g'(x).$$

Equivalently, if we write $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Theorem 3. (Rule for inverses) If $f$ is defined on an interval $I$, and $f'$ exists and is never 0 on $I$, then $f^{-1}$ is differentiable on its domain (the range of $f$), and

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$
Theorem 4. (Rule to rule them all) Given differentiable functions \( f_1(x), \ldots, f_n(x) \) and (possibly negative) numbers \( c_1, \ldots, c_n \), if we set \( f(x) = f_1(x)^{c_1} \cdots f_n(x)^{c_n} \), then
\[
f'(x) = f(x) \left( \frac{c_1f'_1(x)}{f_1(x)} + \cdots + \frac{c_nf'_n(x)}{f_n(x)} \right).
\]

2. DERIVATIVES OF SPECIFIC FUNCTIONS

Theorem 5. Suppose that \( P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \) is a polynomial. Then
\[
f'(x) = nc_nx^{n-1} + (n-1)c_{n-1}x^{n-2} + \cdots + c_1.
\]

Theorem 6. Given \( u \) a differentiable function of \( x \) and \( n \) any real number, then
\[
\frac{d}{dx}u^n = n \cdot \frac{du}{dx} \cdot u^{n-1}.
\]

Theorem 7. Given \( a > 0 \) and \( u \) a differentiable function of \( x \), then
\[
\frac{d}{dx}a^u = \ln a \cdot \frac{du}{dx} \cdot a^u.
\]

In particular,
\[
\frac{d}{dx}e^u = \frac{du}{dx} \cdot e^u.
\]

Theorem 8. Given \( u \) a differentiable function of \( x \), then
\[
\begin{align*}
\frac{d}{dx} \sin u &= \frac{du}{dx} \cdot \cos u, \\
\frac{d}{dx} \cos u &= -\frac{du}{dx} \cdot \sin u, \\
\frac{d}{dx} \tan u &= \frac{du}{dx} \cdot \sec^2 u, \\
\frac{d}{dx} \cot u &= -\frac{du}{dx} \cdot \csc^2 u, \\
\frac{d}{dx} \sec u &= \frac{du}{dx} \cdot \sec u \tan u, \\
\frac{d}{dx} \csc u &= -\frac{du}{dx} \cdot \csc u \cot u.
\end{align*}
\]

Theorem 9. If \( u \) is a differentiable function of \( x \), and \( a > 0 \) a number, then
\[
\frac{d}{dx} \log_a u = \frac{du}{dx} \cdot \frac{1}{u \ln a}.
\]

In particular,
\[
\frac{d}{dx} \ln u = \frac{du}{dx} \cdot \frac{1}{u}.
\]

Theorem 10. If \( u \) is a differentiable function of \( x \), we have:
\[
\begin{align*}
\frac{d}{dx} \sin^{-1} u &= \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}, \\
\frac{d}{dx} \cos^{-1} u &= -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}, \\
\frac{d}{dx} \tan^{-1} u &= \frac{1}{1+u^2} \cdot \frac{du}{dx}, \\
\frac{d}{dx} \cot^{-1} u &= -\frac{1}{1+u^2} \cdot \frac{du}{dx}, \\
\frac{d}{dx} \sec^{-1} u &= \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}, \\
\frac{d}{dx} \csc^{-1} u &= -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}.
\end{align*}
\]
3. Topics and methods

- Terminology/notation relating to derivatives and higher derivatives.
- Choosing which rule to use.
- Terminology for derivatives as rates of change.
- Implicit differentiation: first and higher derivatives.
- Tangent and normal lines. Horizontal tangents.
- Logarithmic differentiation (can alternatively use “rule to rule them all” above, at least when exponents are constants).

Good luck on the exam!