Math 21M Brian Osserman Practice Exam 1 Solutions

These solutions are intended to indicate roughly how much you would be expected to write. Comments in [square brackets] are additional and would not be required. There may be various ways of writing valid explanations for conceptual answers.

1 (14 pts.) Determine whether or not the following limits exist, and calculate them. If the limit does not exist as a number, state whether or not it can be written as ∞ or $-\infty$.

(a) $\lim_{x\to 1^+} \frac{2-x^2}{1-x}$

Answer: $\lim_{x\to 1^+}(2-x^2) = 1$. If we set $u = 1-x$, then as x approaches 1 from above, u approaches 0 from below [above and below switch because the coefficient of x in the formula for u is negative], so

$$\lim_{x\to 1^+} \frac{2-x^2}{1-x} = \left(\lim_{x\to 1^+} 2-x^2\right)\left(\lim_{x\to 1^+} \frac{1}{1-x}\right) = 1\cdot\left(\lim_{u\to 0^-} \frac{1}{u}\right) = -\infty.$$

(b) $\lim_{x\to\infty} \sin x$

Answer: $\lim_{x\to\infty} \sin x$ does not exist, and cannot be written as ∞ or $-\infty$. This is because no matter how big you choose M, one can always find $x_1, x_2 > M$ such that $\sin x_1 = 1$ but $\sin x_2 = -1$. [If $\epsilon < 1$, there is no number L with both 1 and -1 within ϵ of L. The limit isn't $\pm\infty$ since $\sin x$ is bounded between -1 and 1, or more formally, if $N \geq 1$, then $\sin x \leq N$ for all x, so the limit isn't ∞, but if $N \leq -1$, then $\sin x \geq N$ for all x, so the limit isn't $-\infty$.

Also notice that by substitution, this limit is the same as $\lim_{t\to 0^+} \sin(1/t)$, which we used as an example in class with no limit.]

2 (20 pts.) Consider the function

$$f(x) = \begin{cases} \frac{x^3-2x^2+2x-4}{x^2-3x+2} : & x \neq 1,2 \\ 4 : & x = 1,2 \end{cases}$$

At which points is this function continuous, and at which points is it discontinuous? For each discontinuity, say whether or not it is removable. Find all asymptotes to the graph. Sketch the graph.

Answer: The denominator is 0 only at $x = 1, 2$, so $f(x)$ is continuous away from these points. To analyze $x = 1, 2$, we first factor. The denominator factors as $(x-1)(x-2)$. Plugging in $x = 1, 2$, we see the numerator is 0 at $x = 2$, so we can factor it $(x-2)(x^2+2)$. Away from $x = 2$, we can cancel and $f(x)$ is the same as $g(x) = \frac{x^2+2}{x-1}$. $g(x)$ is continuous at $x = 2$, so we see that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = g(2) = 6.$$

But $f(2) = 4$, so $f(x)$ is not continuous at $x = 2$. This is a removable discontinuity, since the limit does exist. On the other hand,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = \lim_{x\to 1}(x^2+2)\cdot\frac{1}{x-1}.$$

Now, $\lim_{x\to 1}(x^2+2) = 3$, and $\lim_{x\to 1^+}\frac{1}{x-1} = \infty$, so $\lim_{x\to 1^+} f(x) = \infty$. But $\lim_{x\to 1^-}\frac{1}{x-1} = -\infty$, so $\lim_{x\to 1^-} f(x) = -\infty$, and we conclude that $\lim_{x\to 1} f(x)$ does not exist. Thus, $f(x)$ is discontinuous at $x = 1$, and the discontinuity is not removable.

The above calculation shows that there is a vertical asymptote at $x = 1$, and this is the only vertical asymptote. For horizontal and oblique asymptotes, look at $f(x)$ as $x \to \pm\infty$. Using $f(x) = g(x)$ for $x \neq 2$, look instead at $\frac{x^2+2}{x-1} = x + 1 + \frac{3}{x-1}$. Now, $\lim_{x\to\infty}\frac{3}{x-1} = 0$, and also $\lim_{x\to-\infty}\frac{3}{x-1} = 0$, so the line $y = x + 1$ is an oblique asymptote for $f(x)$ in both directions.

[Graph omitted; don't worry about making it too accurate, but just include the discontinuities and asymptotes.]

3 (14 pts.) Suppose you are making a square roadsign, with side length x. If your customer wants the sign's area to be within ϵ of 9 square feet, how close does x have to be to 3 feet?

Answer: Given $\epsilon > 0$, start with $|x^2 - 9| < \epsilon$. [We want to replace this with inequalities in terms of $x - 3$.] We do:

$$
\begin{aligned}
-\epsilon < x^2 - 9 < \epsilon \\
9 - \epsilon < x^2 < 9 + \epsilon \\
\sqrt{9-\epsilon} < x < \sqrt{9+\epsilon} \quad (\text{if } \epsilon \le 9) \\
\sqrt{9-\epsilon} - 3 < x - 3 < \sqrt{9+\epsilon} - 3
\end{aligned}
$$

So when $\epsilon \le 9$, we can choose δ to be the smaller of $3 - \sqrt{9-\epsilon}$ and $\sqrt{9+\epsilon} - 3$. [For $\epsilon > 9$, $\delta = \min\{3, \sqrt{18} - 3\}$ works, but that's absurd in context]

4 (8 pts.) Does the equation $x^3 - x - 1 = 0$ have a solution in the interval $[1, 2]$? Explain.

Answer: Yes. It is continuous because it is a polynomial. It is equal to -1 at $x = 1$, and to 5 at $x = 2$, so by the Intermediate Value Theorem it must be equal to 0 somewhere in between 1 and 2.

5 (20 pts.) For each question, answer only "true" or "false". There is no partial credit.

(a) If $f(x) > g(x)$ on $(0,1)$, and both $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^+} g(x)$ exist, then

$$\lim_{x\to 0^+} f(x) > \lim_{x\to 0^+} g(x).$$

Answer: False. [For instance, consider $f(x) = 2x$ and $g(x) = x$.]

(b) If $\lim_{x\to 0^+} f(x)$ is a positive number, then there is some interval $(0,c)$ for $c > 0$ on which $f(x)$ is positive.

Answer: True. [If the limit is L, choose ϵ less than L, and set $c = \delta$.]

(c) The function $f(x) = |x|/x$ has a removable discontinuity at $x = 0$.

Answer: False. [The limit doesn't exist – it is 1 on the right but -1 on the left.]

(d) For every function $f(x)$ such that $\lim_{x\to 0} f(x) = 0$ and every $g(x)$ such that $\lim_{x\to 0} g(x) = \infty$, the limit $\lim_{x\to 0} f(x)\cdot g(x)$ does not exist.

Answer: False. [For instance, consider $f(x) = x^2$ and $g(x) = 1/x$]

6 (10 pts.) If $f(x) = 1/x$, find $f'(2)$ directly from the definition of the derivative.

Answer:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{1/(2+h) - 1/2}{h} \\ &= \lim_{h \to 0} \frac{(2-(2+h))/2(2+h)}{h} \\ &= \lim_{h \to 0} \frac{-h/2(2+h)}{h} \\ &= \lim_{h \to 0} \frac{-1}{2(2+h)} = -1/4. \end{aligned}$$

7 (14 pts.) Suppose $f(x) = x^3 - ax + b$. Find a, b if $(1,1)$ is on the graph $y = f(x)$, and the tangent line to the graph at $(1,1)$ has slope -3.

Answer: If $(1,1)$ is on the graph, then we have $f(1) = 1$. This means $1^3 - a \cdot 1 + b = 1$, so $a = b$. Then $f'(x) = 3x^2 - a$, so $f'(1) = 3 - a$. If $f'(1) = -3$, then $3 - a = -3$, so $a = 6$, and then also $b = 6$.