1 (8 pts.) Calculate $\lim_{x\to 0^+} \frac{\ln(x^2+2x)}{\ln x}$. If it does not exist, state whether or not it is equal to ∞ or $-\infty$.

Answer: This is $-\infty/-\infty$, so using l'Hopital's rule gives

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(x^2+2x)}{\ln x} &= \lim_{x\to 0^+} \frac{(2x+2)/(x^2+2x)}{1/x} \\ &= \lim_{x\to 0^+} \frac{2x+2}{x+2} \\ &= \frac{2}{2} = 1.\end{aligned}$$

2 (17 pts.) A curved road is shaped like a parabola $y = x^2$, where the positive y direction is north and the positive x direction is east. A car is driving northeast on the road, starting from the southernmost point, and an observer is standing at the southernmost point of the road and watching the car, always facing directly towards it. At the moment that the car is 2 miles east of the southernmost point, its motion along the road is moving it further east at a rate of 15 miles per hour. At this moment, how fast is the observer turning (in radians per hour) in order to watch the car?

Answer: Say the position of the car is (x, y), with $(0, 0)$ being the southernmost point on the road, and the angle of the observer from the x-axis is θ. These are all functions of t, and we have that they are related by $y = x^2$ and $\tan\theta = y/x$. We are also given that at the moment in question, $x = 2$, and $\frac{dx}{dt} = 15$. We want to find $\frac{d\theta}{dt}$. Since $y/x = x$, we have $\tan\theta = x$. There are two ways to finish the problem. We could take $\frac{d}{dt}$ on both sides first, which gives $\frac{d\theta}{dt} \cdot \sec^2(\theta) = \frac{dx}{dt}$. We then need to find $\sec^2(\theta)$. We have $\cos\theta = x/\sqrt{x^2+y^2}$, so

$$\sec^2(\theta) = \frac{x^2+y^2}{x^2} = \frac{x^2+x^4}{x^2} = 1 + x^2.$$

Plugging in $x = 2$ and $\frac{dx}{dt} = 15$ we find $\sec^2(\theta) = 5$ and $\frac{d\theta}{dt} \cdot 5 = 15$, so $\frac{d\theta}{dt} = 3$.

Alternatively, if we write $\theta = \tan^{-1}(x)$ and then take $\frac{d}{dt}$, we get

$$\frac{d\theta}{dt} = \frac{dx}{dt} \cdot \frac{1}{1 + x^2} = 15/5 = 3.$$

3 (25 pts.) What are the dimensions of the open-top cylindrical can with the smallest surface area that will hold 1000 cubic cm of liquid?

Answer: Let r be the radius and h the height in cm, and S the surface area in cm^2. Since the volume is assumed to be 1000 cubic cm, we have

$$\pi r^2 h = 1000.$$

On the other hand, since the can is open-top, we have

$$S = 2\pi r h + \pi r^2.$$

Solving for h in terms of r gives

$$h = \frac{1000}{\pi r^2},$$

and substituting gives

$$S = \frac{2000}{r} + \pi r^2.$$

The domain of this is $r > 0$. Then

$$\frac{dS}{dr} = \frac{-2000}{r^2} + 2\pi r.$$

This is defined everywhere, so critical points are where it is 0, or $\frac{2000}{r^2} = 2\pi r$, so $r = \frac{10}{\sqrt[3]{\pi}}$, so if A has an absolute minimum, it must occur here. On the other hand, $\lim_{r \to 0^+} \frac{2000}{r} + \pi r^2 = \infty$ and $\lim_{r \to \infty} \frac{2000}{r} + \pi r^2 = \infty$, so A does have an absolute minimum, at $r = \frac{10}{\sqrt[3]{\pi}}$.

Here, we have $h = \frac{1000}{\pi(100/\pi^{2/3})} = \frac{10}{\sqrt[3]{\pi}}$ as well.

4 (10 pts.) Is it true that for every a, b, the inequality $|\sin b - \sin a| \leq |b - a|$ holds? Why or why not? (Hint: what does the Mean Value Theorem tell you?)

Answer: Yes, it's true: it's certainly true if $a = b$, and if $a \neq b$ and we take $f(x) = \sin x$, the Mean Value Theorem says that there is some c in between a and b with $\frac{f(b)-f(a)}{b-a} = f'(c)$. Taking absolute values and plugging in $f(x) = \sin x$, we get

$$\frac{|\sin b - \sin a|}{|b - a|} = |\cos c| \leq 1,$$

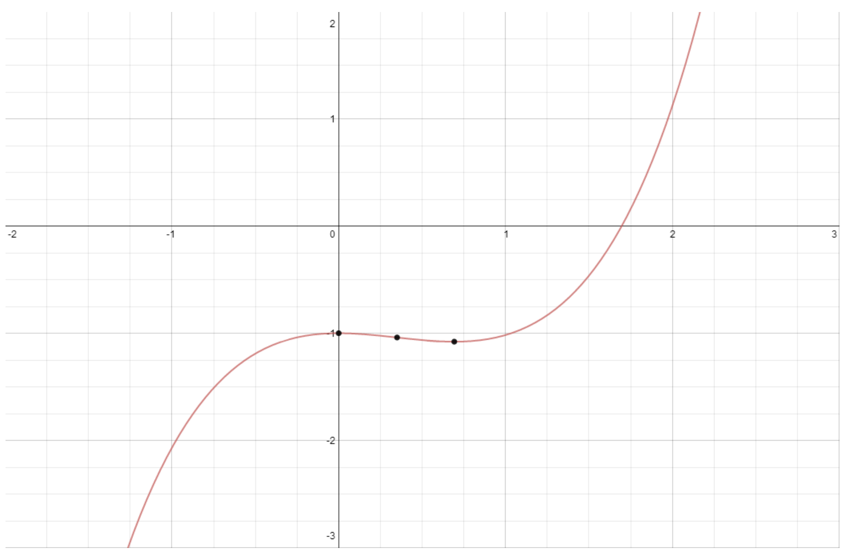
so multiplying through by $|b - a|$ gives the inequality.

5 (25 pts.) Let $f(x) = e^x - 2e^{-x} - 3x$. Find the critical points, and the intervals on which $f(x)$ is increasing and decreasing. Find the inflection points, and the intervals on which the graph is concave up and concave down. Sketch the graph of $f(x)$.

Answer: $f'(x) = e^x + 2e^{-x} - 3$, which is defined everywhere, so critical points are where $f'(x) = e^x + 2e^{-x} - 3 = 0$. Multiplying both sides by e^x (which is always nonzero) gives $0 = (e^x)^2 - 3e^x + 2 = (e^x - 1)(e^x - 2)$, so the critical points are where $e^x = 1$ or $e^x = 2$, which means at $x = 0$ and $x = \ln 2$. By plugging in values or direct analysis, $f'(x)$ is positive on $(-\infty, 0)$ and $(\ln 2, \infty)$ and negative on $(0, \ln 2)$, so $f(x)$ is increasing on $(-\infty, 0)$ and $(\ln 2, \infty)$ and decreasing on $(0, \ln 2)$.

$f''(x) = e^x - 2e^{-x}$, which is 0 if $e^x = 2e^{-x}$, so $(e^x)^2 = 2$, and $x = \ln\sqrt{2} = \frac{\ln 2}{2}$. Again by testing values or direct analysis, the graph is concave up on $(\frac{\ln 2}{2}, \infty)$ and concave down on $(-\infty, \frac{\ln 2}{2})$, so we also see that $x = \frac{\ln 2}{2}$ is an inflection point.

The graph is as follows.



6 (15 pts.) Suppose $u(x)$ is a differentiable function of x, and let $f(x) = x^{u(x)}$, for $x > 0$. Using logarithmic differentiation, find $f'(x)$, in terms of $x, u(x)$, and $u'(x)$.

Answer: Taking ln on both sides,

$$\ln f(x) = \ln x^{u(x)} = u(x) \ln x.$$

Taking $\frac{d}{dx}$,

$$f'(x)/f(x) = u'(x) \ln x + u(x)/x.$$

So

$$f'(x) = f(x)(u'(x) \ln x + \frac{u(x)}{x}) = x^{u(x)}(u'(x) \ln x + \frac{u(x)}{x}).$$