Math 21M Brian Osserman Practice Exam 3

Give yourself 1 hour to take this exam. Be sure to fully justify all your answers.

1 (30 pts.) Evaluate the following integrals (consider them as improper integrals where necessary, and state clearly when you are doing so):

(a) $\displaystyle\int_2^3 \frac{x^2}{(x-1)(x^2+2x+1)}\,dx.$

Solution: Use partial fractions: first, factor $x^2+2x+1=(x+1)^2$. Then, solve

$$\begin{aligned}\frac{x^2}{(x-1)(x+1)^2} &= \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\\ &= \frac{A(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)(x+1)^2}.\end{aligned}$$

Plugging $x=1$ into the numerators gives $1=4A$, so $A=1/4$. Plugging in $x=-1$ gives $1=-2C$, so $C=-1/2$. Plugging in $x=0$ gives $0=A-B-C$, so $B=A-C=1/4-(-1/2)=3/4$. So

$$\begin{aligned}\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} &= \int \frac{1/4}{x-1}+\frac{3/4}{x+1}-\frac{1/2}{(x+1)^2}\,dx\\ &= 1/4\ln|x-1|+3/4\ln|x+1|+1/2\frac{1}{x+1}+K.\end{aligned}$$

(b) $\displaystyle\int \frac{7\,dx}{(9x^2+1)^2}.$

Solution: Use the trig substitution $x=\frac{1}{3}\tan\theta$, then $9x^2+1=\sec^2\theta$ and $dx=\frac{1}{3}\sec^2\theta\,d\theta$, so

$$\begin{aligned}\int \frac{7\,dx}{(9x^2+1)^2} &= \frac{7}{3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta\\ &= \frac{7}{3}\int \cos^2\theta\,d\theta\\ &= \frac{7}{6}\int 1+\cos 2\theta\,d\theta\\ &= \frac{7}{6}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C\\ &= \frac{7}{6}\left(\tan^{-1}(3x)+\frac{1}{2}\sin(2\tan^{-1}(3x))\right)+C.\end{aligned}$$

(c) $\int_0^2 \frac{x+1}{\sqrt{4-x^2}}\, dx.$

Solution: Note that this is an improper integral, since the integrand goes to ∞ as x approaches 2. We first integrate

$$\int_0^b \frac{x+1}{\sqrt{4-x^2}}\, dx = \int_0^b \frac{x}{\sqrt{4-x^2}}\, dx + \int_0^b \frac{1}{\sqrt{4-x^2}}\, dx,$$

and using the substitution $u = 4 - x^2$ for the first integral, we get

$$\begin{aligned} \int_4^{4-b^2} \frac{-1}{2\sqrt{u}}\, du + \sin^{-1} \frac{x}{2}\Big]_0^b &= -\sqrt{u}\Big]_4^{4-b^2} + \sin^{-1}\frac{b}{2} - \sin^{-1}(0) \\ &= -\sqrt{4-b^2} + \sqrt{4} + \sin^{-1}(b/2) \\ &= -\sqrt{4-b^2} + 2 + \sin^{-1}(b/2). \end{aligned}$$

If we then take the limit as $b \to 2$, we get

$$-\sqrt{0} + 2 + \sin^{-1}(1) = 2 + \frac{\pi}{2}.$$

2 (20 pts.) The region enclosed by the curves

$$y = x^{1/2}, \quad y = x^{1/4}, \quad 0 \le x \le 1$$

is revolved about the y-axis. Find the volume of the resulting solid.

Solution: We use cylindrical shells (but washers would also work, if you solve for x in terms of y). For $0 \le x \le 1$ we have $x^{1/4} \ge x^{1/2}$, so the height of a shell is given by $x^{1/4} - x^{1/2}$. Since we revolve about the y-axis, the radius is x, so the volume is

$$\begin{aligned} 2\pi \int_0^1 x(x^{1/4} - x^{1/2}) dx &= 2\pi \int_0^1 (x^{5/4} - x^{3/2}) dx \\ &= 2\pi \left(\frac{4}{9} x^{9/4} - \frac{2}{5} x^{5/2} \Big]_0^1 \right) \\ &= 2\pi \left(\frac{4}{9} - \frac{2}{5} - 0 \right) \\ &= \frac{4}{45}\pi. \end{aligned}$$

3 (20 pts.) The shape of a water tank is described by revolving the curve $y = x^2$, $0 \le x \le 4$, about the y-axis (x, y are in meters). The tank is filled with water. What is the work required to pump all the water out over the top edge of the tank? (The weight-density of water is $10,000$ N/m^3)

Solution: If we take a horizontal slice of water at height y, with thickness dy, we get a disk of radius $x = \sqrt{y}$. Since x goes from 0 to 4, we see that y goes from 0 to 16, so the tank has height 16, and we need to lift this slice a distance of $16 - y$ m. The weight of the slice in Newtons is

$$10,000 \cdot \pi(\sqrt{y})^2 dy = 10,000\pi y dy.$$

Thus, the work to pump out the slice is $10,000\pi y(16 - y)dy$ N·m. To find the total work to pump the water out, we have to integrate over all slices, so we get (in N·m):

$$\begin{aligned} \int_0^{16} 10,000\pi y(16-y)dy &= 10,000\pi \left(8y^2 - \frac{1}{3}y^3 \Big]_0^{16} \right) \\ &= 10,000\pi \left(2048 - \frac{4096}{3} - 0 \right) \\ &= \frac{20,480,000}{3}\pi. \end{aligned}$$

4 (10 pts.) Does the following improper integral converge or diverge?

$$\int_2^\infty \frac{x\ dx}{\sqrt{x^4 - 1}}.$$

Solution: We use the comparison test. For $x \ge 2$, we have $x^4 - 1 < x^4$, so $\sqrt{x^4 - 1} < \sqrt{x^4} = x^2$, so $1/\sqrt{x^4 - 1} > 1/x^2$, and finally $x/\sqrt{x^4 - 1} > x/x^2 = 1/x$. But $\int_2^\infty 1/x\ dx = \lim_{b\to\infty}(\ln b - \ln 2)$ diverges, so by the comparison test, $\int_2^\infty \frac{x\ dx}{\sqrt{x^4-1}}$ also diverges.

5 (10 pts.) Find the general solution of the following differential equation:

$$(\sec x)\frac{dy}{dx} = e^{y+\sin x}.$$

Solution: We can rewrite as $\frac{1}{\cos x}\frac{dy}{dx} = e^y \cdot e^{\sin x}$, and then put the y terms on the left and the x terms on the right to get

$$e^{-y}dy = \cos x e^{\sin x}dx.$$

Taking indefinite integrals (using the substitution $u = \sin x$ on the right) gives

$$-e^{-y} = e^{\sin x} + C,$$

and negating and then taking ln on both sides gives

$$y = -\ln(-e^{\sin x} - C).$$

6 (10 pts.) Use the Max-Min inequality to find upper and lower bounds for $\int_0^1 \frac{1}{1+x^2}\,dx$. Then do the same thing, but first breaking up the integral as

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{1/2} \frac{1}{1+x^2}\,dx + \int_{1/2}^1 \frac{1}{1+x^2}\,dx.$$

Can you relate what you've done to Riemann sums?

Solution: Note that because $1+x^2$ is increasing for $x \geq 0$, $f(x) = \frac{1}{1+x^2}$ is decreasing, so on $[0,1]$, the maximum value of $f(x)$ is 1 (at $x=0$) and the minimum value of $f(x)$ is $1/2$ (at $x=1$). The Max-Min inequality then says that

$$(1-0)\frac{1}{2} = \frac{1}{2} \leq \int_0^1 \frac{1}{1+x^2}\,dx \leq (1-0)\cdot 1 = 1.$$

Now, if we break up the integral, on $[0,1/2]$ the maximum is still 1, but now the minimum is $1/(1+1/4) = 4/5$, at $x = 1/2$. On $[1/2,1]$, the maximum is now $4/5$, and the minimum is still $1/2$. Combining the Max-Min inequalities for the two integrals gives

$$\begin{aligned}(1/2-0)\frac{4}{5} + (1-1/2)\frac{1}{2} = 13/20 \leq \int_0^{1/2} \frac{1}{1+x^2}\,dx + \int_{1/2}^1 \frac{1}{1+x^2}\,dx \\ \leq (1/2-0)\cdot 1 + (1-1/2)\frac{4}{5} = 9/10.\end{aligned}$$

So we get $13/20 \leq \int_0^1 \frac{1}{1+x^2}\,dx \leq 9/10$.

This can be reinterpreted as calculating the upper and lower Riemann sums for the integral, first using the partition of $[0,1]$ into one piece, and then using the partition into the two pieces $[0,1/2], [1/2,1]$. Since we know that the actual integral is always bounded between any lower sum and any upper sum, we could have justified our bounds this way instead of using the Max-Min inequality.