CHEVALLEY'S THEOREM AND COMPLETE VARIETIES

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In this note, we introduce the concept which plays the role of compactness for varieties – completeness. We prove that completeness can be characterized in terms of existence of extensions of morphisms from nonsingular curves, and conclude that projective varieties are complete. As a prelude to this, we also prove Chevalley's theorem on images of morphisms.

1. Chevalley's theorem

We have already seen that the image of a morphism of a variety need not be a subvariety (that is, it need not be a closed subset of an open subset). We recall the example:

Example 1.1. Consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ determined by $(x, y) \mapsto (x, xy)$. Its image is $\mathbb{A}^2 \smallsetminus Z(x) \cup \{(0, 0)\}$.

Chevalley's theorem asserts that this example is typical: the image of a morphism is always a finite union of subvarieties.

Theorem 1.2 (Chevalley)**.** *Let $\varphi : X \to Y$ be a morphism of varieties. Then $\varphi(X)$ can be written as a finite union of (not necessarily closed) subvarieties of Y.*

Note that $\varphi(X)$ must be irreducible, so there is a unique subvariety of Y contained in $\varphi(X)$ which is dense in $\varphi(X)$; the other subvarieties of the theorem are all in its closure. The key statement to prove is:

Proposition 1.3. *If $\varphi : X \to Y$ is a dominant morphism of prevarieties, then $\varphi(X)$ contains an open subset of Y.*

In the proof we will need the following statement, which isn't hard to prove, and we already used implicitly in our results on normalizations.

Proposition 1.4. *Let $\varphi : X \to Y$ be a dominant morphism of affine varieties which makes $A(X)$ into an integral extension of $A(Y)$. Then φ is surjective.*

Although the proof is not difficult, we (continue to) omit it. The next step is to show that any dominant morphism factors as a composition of two particular types of dominant morphisms, which will be easier to analyze.

Lemma 1.5. *If $\varphi : X \to Y$ is a dominant morphism of affine varieties, and r is the transcendence degree of the induced field extension $K(X)/K(Y)$, then φ factors as a composition of dominant morphisms $X \to Y \times \mathbb{A}^r \to Y$, where the second morphism is the projection morphism.*

Proof. Let $f_1, \dots, f_m$ be generators of $A(X)$ over $A(Y)$. Then the f_i also generate $K(X)$ over $K(Y)$, so we can reorder indices such that $f_1, \dots, f_r$ are algebraically independent over $K(Y)$. Let $R = A(Y)[f_1, \dots, f_r] \subseteq A(X)$. Since the f_i are algebraically independent over $K(Y)$ they are algebraically independent over $A(Y)$, so R is isomorphic to an r-variable polynomial ring, which is to say that $R \cong A(Y \times \mathbb{A}^r)$. Then the inclusions $A(Y) \hookrightarrow R \hookrightarrow A(X)$ induce the desired factorization. □

We can now prove that the image of a dominant morphism contains an open subset.

Proof of Proposition 1.3. Let V be an affine open subset of Y, and U an affine open subset of X such that $\varphi(U) \subseteq V$. Then it clearly suffices to prove $\varphi(U)$ contains an open subset of V, so we have reduced to the affine case. Applying Lemma 1.5, it suffices to prove that the image of X in $Y \times \mathbb{A}^r$ contains an open subset, and that the projection morphism $Y \times \mathbb{A}^r \to Y$ is open.

For the first assertion, we have that the transcendence degrees of $K(X)$ and $K(Y \times \mathbb{A}^r)$ are equal, so the morphism makes $K(X)$ into an algebraic extension of $K(Y \times \mathbb{A}^r)$. Suppose $f_1, \ldots, f_m$ generate $A(X)$ over $A(Y \times \mathbb{A}^r)$. Then each f_i is a root of some polynomial $g_i = \sum_j c_{i,j} t^j$ over $K(Y \times \mathbb{A}^r)$, which can assume to be monic. Let h be the product over all i, j of the denominators of $c_{i,j}$ (considering $K(Y \times \mathbb{A}^r)$ as the fraction field of $A(Y \times \mathbb{A}^r)$). We have the induced morphism of open subsets $X_h \to (Y \times \mathbb{A}^r)_h$, and we see that $A(X_h) = A(X)_h$ is still generated by the f_i over $A((Y \times \mathbb{A}^r)_h) = A(Y \times \mathbb{A}^r)_h$. But each f_i is integral over $A(Y \times \mathbb{A}_r)_h$ by construction, so by Proposition 1.4, we have that X_h surjects onto $(Y \times \mathbb{A}^r)_h$, and thus the image of X contains the open subset $(Y \times \mathbb{A}^r)_h \subseteq Y \times \mathbb{A}^r$.

It remains to prove that for any $U \subseteq Y \times \mathbb{A}^r$ open, the image of U under the projection morphism $Y \times \mathbb{A}^r \to Y$ is open (we only need to prove it contains an open subset, but the stronger statement is no harder). Any such U is a union of open subsets of the form $(Y \times \mathbb{A}^r)_f$, for some nonzero $f \in A(Y \times \mathbb{A}^r) = A(Y)[t_1, \ldots, t_r]$, so we may thus assume that U is of this form. Let $I \subseteq A(Y)$ be the ideal generated by the coefficients of f. We claim that the image of U is precisely $A(Y) \smallsetminus Z(I)$. Indeed, given $Q \in Y$, if we let Z_Q be the preimage of Q in $Y \times \mathbb{A}^r$, we have $Z_Q \cong \mathbb{A}^r$, and the inclusion $Z_Q \to Y \times \mathbb{A}^r$ is induced by the ring homomorphism $A(Y)[t_1, \ldots, t_r] \to k[t_1, \ldots, t_r]$ obtained by sending $g \in A(Y)$ to $g(Q)$. We thus see that $f|_{Z_Q}$ is obtained by evaluating the coefficients of f at Q, and so $Z_Q \cap U = Z_Q \smallsetminus Z(f)$ is non-empty if and only if f remains nonzero when its coefficients are evaluated at Q, which is precisely equivalent to the condition that $Q \notin I$. But Q is in the image of U if and only if $Z_Q \cap U \neq \emptyset$, so we conclude that the image of U is the complement of $Z(I)$, as desired. □

Chevalley's theorem then follows easily.

Proof of Theorem 1.2. Given $\varphi : X \to Y$, let $Z \subseteq Y$ be the closure of $\varphi(X)$; our proof is by induction on $\dim(Z)$. If $\dim(Z) = 0$, then φ is constant and there is nothing to show. Otherwise, assume $\dim(Z) = d > 0$, and we have the theorem already for dimensions smaller than d.

Now, we have a dominant morphism $X \to Z$, so let $U \subseteq Z$ be the maximal open subset of Z contained in $\varphi(X)$; this exists by Proposition 1.3. Then let $Z' = Z \smallsetminus U$; this has dimension less than d, and is not necessary a variety, but can be write it as a finite union of varieties $Z_1, \ldots, Z_m$. Similarly, $\varphi^{-1}(Z_i)$ is not necessarily a variety, but can be written as a finite union of varieties $X_{i,1}, \ldots, X_{i,m_i} \subseteq X$. We have

$$\begin{aligned}\varphi(X) &= U \cup \varphi(\varphi^{-1}(Z_1)) \cup \cdots \cup \varphi(\varphi^{-1}(Z_m)) \\ &= U \cup \bigcup_{i,j} \varphi(X_{i,j}),\end{aligned}$$

and by the induction hypothesis each $\varphi(X_{i,j})$ is a finite union of subvarieties of $Z_i \subseteq Y$, so we conclude the theorem. □

2. Completeness and limits

We now apply our discussion of curves to give a definition for varieties which is analogous to the notion of compactness for topological spaces. We have the opposite problem that we had with the Hausdorff condition: every variety is compact in the Zariski topology, because its underlying topological space is Noetherian. However, the fix is the same as before: we give a rephrasing of the compactness condition in topology which will turn out to agree better with our intuition when we apply it to varieties.

Exercise 2.1. A topological space X is compact if and only if for every topological space Y, the projection map $X \times Y \to Y$ is a closed map.

Motivated by this, we define:

Definition 2.2. A variety X is **complete** if for all varieties Y, the projection morphism $X \times Y \to Y$ is a closed map.

Remark 2.3. One can apply the definition of completeness to prevarieties as well, but it is traditional to reserve the term "complete" for varieties. This is related to the French tradition that compactness should incorporate the Hausdorff condition as well.

One reason for this definition is its relation to closed morphisms:

Proposition 2.4. *If X is complete, any closed subvariety of X is complete.*
If we also have Y an arbitrary variety, and $\varphi : X \to Y$ a morphism, then φ is closed.

Proof. The first assertion is immediate from the definition, since if Z is closed in X, we have $Z \times Y$ closed in $X \times Y$ for any Y.

For the second assertion, let $\Gamma = \{(x, \varphi(x) : x \in X\} \subseteq X \times Y$ be the graph of φ. We can express Γ as the preimage of the diagonal $\Delta(Y) \subseteq Y \times Y$ under the morphism $\varphi \times \mathrm{id} : X \times Y \to Y \times Y$. Since Y is a variety, $\Delta(Y)$ is closed, so Γ is closed. But $\varphi(X)$ is precisely the image of Γ under the projection $X \times Y \to Y$, so we conclude from the completeness hypothesis on X that $\varphi(X)$ is closed. □

Remark 2.5. This is a natural property for complete varieties to have, since a continuous map from a compact topological space to a Hausdorff space is closed. In fact, this property characterizes complete varieties – Nagata proved that every variety can be realized as an open subset of a complete variety, so in particular if X is not complete, the inclusion as an open subset of a complete variety is not a closed mapping. However, the proof of Nagata's theorem is beyond the scope of this course.

We wish to give a more intuitive necessary and sufficient criterion for completeness. Because the ideas are closely related, we will also give a more intuitive criterion for a prevariety to be a variety. In the informal language of limits we introduced, we will show that a prevariety is a variety if and only if limits are unique when they exist, and that a variety is complete if and only if limits always exist.

An important lemma is the following:

Lemma 2.6. *Suppose $\varphi : X \to Y$ is a morphism of prevarieties, and $Q \in Y$ is in the closure of $\varphi(X)$. Then there exists a nonsingular curve C and $P \in C$, and morphisms $\tilde{\psi} : C \smallsetminus \{P\} \to X$ and $\psi : C \to Y$ such that $\varphi \circ \tilde{\psi} = \psi|_{C \smallsetminus \{P\}}$, and and $\psi(P) = Q$.*

An intermediate lemmas is the following.

Lemma 2.7. *Suppose X is a prevariety, U a nonempty open subset, and $P \in X \smallsetminus U$. Then there exists a subprevariety Z of X which is a curve, such that $P \in Z$, and $U \cap Z \neq \emptyset$.*

Proof. It suffices to produce such a Z inside any open neighborhood of P in X, so let $V \subseteq X$ be an affine open neighborhood of Q. We prove the statement by induction on $\dim X$; if $\dim X = 1$, we simply let $Z = X$. For $\dim X > 1$, let $\mathfrak{m}_P$ be the maximal ideal of $A(V)$ corresponding to P, and let $I \subseteq A(V)$ be the radical ideal with $V \smallsetminus U = Z(I)$. Choose $Q_1, \ldots, Q_r \neq P$ in $Z(I)$, with one Q_i in each irreducible component of $Z(I)$ having codimension 1 (we may have $r = 0$). We claim there exists f such that $f(P) = 0$, but $f(Q_i) \neq 0$ for all i.

We construct f as follows: for each $i \neq j$, first find $f_{i,j}$ such that $f_{i,j}(Q_i) = 0$, but $f_{i,j}(Q_j) \neq 0$. Such an $f_{i,j}$ exists because $\mathfrak{m}_{Q_j} \not\supseteq \mathfrak{m}_{Q_i}$. Also choose g_i such that $g_i(P) = 0$, but $g_i(Q_i) \neq 0$, which exists for the same reason. Let $f_i = g_i \prod_{j \neq i} f_{i,j}$. Then $f_i(Q_i) \neq 0$, but $f_i(Q_j) = f_i(P) = 0$ for all $j \neq i$. We can then let $f \sum_i f_i$. In the case $r = 0$, we can choose any nonzero $f \in \mathfrak{m}_P$. Now, $Z(f)$ a closed subset of X containing P; and not containing any component of $Z(I)$ having codimension 1. Let X' be an irreducible component of $Z(f)$ which contains P. Then $\dim X' = \dim X - 1$, and since $Z(f)$ doesn't contain any component of $Z(I)$ of codimension 1, we find that X' cannot be contained in $Z(I)$, so $X' \cap U \neq \emptyset$. Replacing X by X' and U by $X' \cap U$, we apply the induction hypothesis to find the desired curve. □

We can now prove the first lemma.

Proof of Lemma 2.6. Our first claim is that there is a subprevariety $D \subseteq Y$ which is a curve containing Q, and such that $D \cap \varphi(X)$ is dense in D. But by Proposition 1.3, if Y' is the closure of $\varphi(X)$ in Y, we know that $\varphi(X)$ contains an open subset of Y', so this follows immediately from Lemma 2.7. Now, let $Z \subseteq X$ be the preimage of D under φ. Since $D \cap \varphi(X)$ is dense in D, some irreducible component X' of Z maps dominantly to D. Choose $Q' \in D \cap \varphi(X')$; then $\varphi^{-1}(Q') \cap X'$ is closed in X', and cannot be all of X'. Again using Lemma 2.7, there is a curve C' in X' which meets $\varphi^{-1}(Q') \cap X'$ but is not contained in it. We then see that C' maps dominantly to D, since its image must be irreducible and strictly contains Q'. Let $\tilde{C}$ be the normalization of D inside $K(C')$, and let $P \in \tilde{C}$ be a point mapping to Q. By construction, $K(\tilde{C}) \cong K(C')$, so we get a birational map $\tilde{C} \dashrightarrow C'$ commutes with the given morphisms to D. Let $U \subseteq \tilde{C}$ be the open subset on which the rational map is defined. We can set $C = U \cup \{P\}$, which is still an open subset of $\tilde{C}$, and we obtain morphisms $C \to D \subseteq Y$ and $C \smallsetminus \{P\} \to C' \subseteq X' \subseteq X$ satisfying the desired conditions. □

Before stating the theorem, we give one more lemma.

Lemma 2.8. *Let $\varphi : C \to D$ be a birational morphism of curves, with D nonsingular. Then φ is an isomorphism of C onto the open subset $\varphi(C)$ of D.*

Proof. Let $\nu : \tilde{C} \to C$ be the normalization, and let $\bar{C}$ and $\bar{D}$ be the nonsingular projective curves having $\tilde{C}$ and D as open subsets, respectively. Then $\varphi \circ \nu$ induces a birational map $\bar{C} \to \bar{D}$, which is necessarily an isomorphism. But then both $\tilde{C}$ and D are identified as open subsets of a given curve, compatibly with the morphism $\varphi \circ \nu$, so we conclude that $\varphi \circ \nu$ must map $\tilde{C}$ isomorphically onto an open subset of D. By the surjectivity of ν, this open subset is $\varphi(C)$, and then the morphism $\varphi(C) \to \tilde{C} \to C$ show that φ is an isomorphism onto $\varphi(C)$, as desired. □

We now give the promised more intuitive geometric description in terms of algebraic limits of completeness, as well as of the separation property distinguishing varieties from prevarieties. In a related but more general form, this theorem gives what are typically called valuative criteria.

Theorem 2.9. *A prevariety X is a variety if and only if for all nonsingular curves C, and points $P \in C$, and morphisms $\varphi : C \smallsetminus \{P\} \to X$, there is at most one extension of φ to a morphism $C \to X$.*

A variety X is complete if and only if for all nonsingular curves C, and points $P \in C$, and morphisms $\varphi : C \smallsetminus \{P\} \to X$, there exists a (necessarily unique) extension of φ to a morphism $C \to X$.

Proof. For the first statement, we already know that if X is a variety, then the stated condition holds. Conversely, suppose the condition holds, and consider the diagonal morphism $\Delta : X \to X \times X$. Let $Q \in X \times X$ be in the closure of $\Delta(X)$. Then by Lemma 2.6, there is a nonsingular curve C and a point $P \in C$, with morphisms $\tilde{\psi} : C \smallsetminus \{P\} \to X$ and $\psi : C \to X \times X$ such that $\psi(P) = Q$, and $\psi = \Delta \circ \tilde{\psi}$ on $C \smallsetminus \{P\}$. We then get two extensions of $\tilde{\psi}$ to all of C by composing ψ

with the projection morphisms p_1, p_2. By hypothesis, these extensions are unique, so we conclude that $p_1(\psi(P)) = p_2(\psi(P))$, so $Q = \psi(P) \in \Delta(X)$, and $\Delta(X)$ is closed.

Now, suppose X is a complete variety. Given C, the point $P \in C$, and $\varphi : C \smallsetminus \{P\} \to X$, consider the product $X \times C$. Let Z be the closure of $\{(\varphi(P), P) : P \in C\} \subseteq X \times C$. Then we have dominant morphisms $C \smallsetminus \{P\} \to Z \to C$, and the image of Z is closed in C by hypothesis, so we must have $Z \to C$ surjective. By Lemma 2.8, we conclude that $Z \to C$ is an isomorphism. Inverting the isomorphism, it follows that we have a morphism $C \to Z$ extending $C \smallsetminus \{P\} \to Z$, and taking the first projection we get the desired extension of φ to all of C.

Conversely, suppose X satisfying the stated condition. Given any variety Y, let Z be a closed subset of $X \times Y$. Let $Q \in Y$ be in the closure of the image of Z under the projection morphism p_2. By Lemma 2.6, there is a nonsingular curve C and a point $P \in C$, with morphisms $\tilde{\psi} : C \smallsetminus \{P\} \to Z$ and $\psi : C \to Y$ such that $\psi(P) = Q$, and $\psi = p_2 \circ \tilde{\psi}$ on $C \smallsetminus \{P\}$. By hypothesis, we can extend $p_1 \circ \tilde{\psi}$ to a morphism $\psi' : C \to X$, and we see that if we take the product morphism $\psi' \times \psi : C \to X \times Y$, it extends $\tilde{\psi}$, and must therefore have image contained in Z, since Z is closed. Moreover, by definition we have that $Q = p_2((\psi' \times \psi)(P))$, so $Q \in p_2(Z)$, and we conclude $p_2(Z)$ is closed. □

We immediately conclude:

Corollary 2.10. *Any projective variety is complete.*

From Proposition 2.4 we then conclude:

Corollary 2.11. *Any morphism from a projective variety to an arbitrary variety is closed.*

Remark 2.12. There are direct, algebraic proofs of Corollary 2.10. See for instance Theorem 2 of Chapter I, §5.2 of [2]. However, the point of our approach is to show that if one builds up enough foundational tools, one can start to prove interesting results more geometrically, without resorting to going back to definitions and using algebra to prove each result.

Exercise 2.13. In this exercise, we give a proof of Chow's Lemma. It is clear that every complete variety is birational to some projective variety. However, a much stronger statement is true: given a complete variety X, there exists a projective variety X' together with a birational morphism $X' \to X$ (which is necessarily surjective, by Corollary 2.11).

Let $\{U_i\}$ be an affine open cover of X, and let Y_i be the closures of the U_i in projective space. Let U be the intersection of the U_i, and

$$\varphi : U \to X \times Y_1 \times \cdots \times Y_n$$

the morphism induced by the inclusions of U into X and the Y_i. Let X' be the closure of $\varphi(U)$. Let $p_1 : X' \to X$ be the morphism induced by the first projection, and $p_2 : X' \to Y_1 \times \cdots \times Y_n$ be the morphism induced by projection to the remaining factors.

(a) Show that p_1 gives an isomorphism $p_1^{-1}(U) \to U$. Hint: first prove that $\varphi(U)$ is an open subset of X'.

(b) Show that p_2 induces an isomorphism of X' onto a closed subvariety of $Y_1 \times \cdots \times Y_n$. Hint: first prove that

$$X' \cap X \times Y_1 \times \cdots \times U_i \times \cdots \times Y_n = X' \cap U_i \times Y_1 \times \cdots \times Y_n,$$

by considering the projections to X and to Y_i for each.

(c) Conclude Chow's lemma.

We see that even if X is not quasiprojective, it remains true that there are no nonconstant functions which are globally regular on X.

Exercise 2.14. Prove that if X is a complete variety, then $\mathcal{O}(X) = k$.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Igor Shafarevich, *Basic algebraic geometry 1. varieties in projective space*, second ed., Springer-Verlag, 1994.