# DIFFERENTIAL FORMS

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Differentials are an important topic in algebraic geometry, allowing the use of some classical geometric arguments in the context of varieties over any field. We will use them to define the genus of a curve, and to analyze the ramification of morphisms between curves. Although differentials remain important for arbitrary varieties, we will restrict our treatment to the case of nonsingular varieties.

## 1. Differential forms

A differential form is not a function, but can be defined in an analogous manner.

**Definition 1.1.** Let $X$ be a nonsingular variety, and $U \subseteq X$ an open subset. A **differential form** on $U$ associates to each point $P \in U$ an element of the Zariski cotangent space $T^*_P(X)$.

Differential forms as defined above play a role analogous to that of arbitrary functions: we need to restrict to a much smaller collection of them in order to obtain a useful concept. We do this by observing that for every regular function, we have an associated differential form.

**Definition 1.2.** Given $U \subseteq X$ an open subset of a nonsingular variety, and $f \in \mathcal{O}(U)$, the differential form $df$ associated to $f$ is defined as follows: for $P \in U$, let $df(P) \in \mathfrak{m}_P/\mathfrak{m}_P^2$ be the equivalence class of $\langle U, f - f(P)\rangle$.

A differential form $\omega$ on $U$ is **regular** if for every $P \in U$, there exist an open neighborhood $V \subseteq U$ of $P$ and regular functions $f_1, \dots, f_m, g_1, \dots, g_m \in \mathcal{O}(V)$ such that $\omega|_V = \sum_i f_i dg_i$.

*Notation* 1.3. We denote by $\Omega(U)$ the set of regular differential forms on $U$.

It is clear that $\Omega(U)$ is a module over $\mathcal{O}(U)$. Moreover, $d$ defines a map $\mathcal{O}(U) \to \Omega(U)$ which is visibly $k$-linear, but not $\mathcal{O}(U)$-linear. Instead, we have the **Leibniz rule**:

*Exercise* 1.4. Show that for any $f, g \in \mathcal{O}(U)$, we have $d(fg) = fdg + gdf$.

This in turn gives us a chain rule for differential forms.

*Exercise* 1.5. Suppose that $g \in k(t_1, \dots, t_n)$, and $f_1, \dots, f_n$ are regular on $U \subseteq X$. Then away from the zero set of the denominator of $g$, we have

$$d(g(f_1, \dots, f_n)) = \sum_{i=1}^{n} \frac{\partial g}{\partial t_i}(f_1, \dots, f_n) df_i.$$

Because of the nonsingularity hypothesis, locally on $X$ the modules of differential forms are free of rank equal to the dimension of $X$.

**Lemma 1.6.** *Given $P \in X$, if $f_1, \dots, f_n \in \mathcal{O}_P$ give a basis of $\mathfrak{m}_P/\mathfrak{m}_P^2$, there exists an open set $U \ni P$ on which all the $f_i$ are regular, and such that for every open subset $V \subseteq U$, every $\omega \in \Omega(V)$ can be written uniquely as*

$$\sum_i g_i df_i$$

*for some $g_i \in \mathcal{O}(V)$. For every $Q \in U$, we have that the $f_i - f_i(Q)$ give a basis of $\mathfrak{m}_Q/\mathfrak{m}_Q^2$.*

*Proof.* First let $U'$ be an affine open neighborhood of $P$ in $X$ on which all the $f_i$ are regular. Then extending $f_1, \ldots, f_n$ to a set of generators of $A(U')$, we obtain an imbedding $U' \subseteq \mathbb{A}^m$ with coordinates $t_1, \ldots, t_m$ such that $t_i|_{U'} = f_i$ for $i = 1, \ldots, n$. Let $g_1, \ldots, g_d$ be a set of generators of $I(U') \subseteq \mathbb{A}^m$. Then for each $i$, if we restrict to $U'$ we have

$$0 = dg_i = \sum_j \frac{\partial g_i}{\partial t_j} dt_j.$$

Because $X$ is nonsingular at $P$, we have that the rank of the Jacobian matrix $(\partial g_i/\partial t_j(P))$ is equal to $m - n$, and by our hypothesis that the $f_j = t_j|_{U'}$ generate $\mathfrak{m}_P/\mathfrak{m}_P^2$, we find that every $dt_j|_{U'}$ can be expressed in terms of $df_1, \ldots, df_n$, with coefficients that are rational functions on $X$, regular at $P$. If we let $U \subseteq U'$ be an open neighborhood of $P$ on which all the coefficient functions are regular, we claim that for any $V \subseteq U$ open, and $\omega \in \Omega(V)$, there exist unique $g_i \in \mathcal{O}(V)$ with

$$\omega = \sum_i g_i df_i.$$

We observe that at any point $Q \in U'$, the $dt_j|_{U'}$ for $j = 1, \ldots, m$ span $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, and it follows that if $Q \in U'$, in fact the $df_j$ for $j = 1, \ldots, n$ span $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, so they must be a basis. We conclude that the desired $g_i$ are unique, if they exist. On the other hand, since every regular function on any open subset of $U'$ is a rational function in the $t_i$, using Exercise 1.5 we know that $\omega$ can be written locally near any point $Q \in V$ as a sum of the form

$$\sum_i h_i dt_i,$$

where the $h_i$ are rational functions in the $t_j$, regular at $Q$. But we can similarly express each $dt_i$ for $i > n$ as a combination of the $dt_1, \ldots, dt_n$ with coefficients being rational functions in the $t_j$, regular at $Q$, so we obtain the desired express. $\square$

It is then clear that we have:

**Corollary 1.7.** *Given $f_i$ and $U$ as in Lemma 1.6, we have that $\omega = \sum_i g_i df_i$ vanishes at $P \in U$ if and only all the $g_i$ vanish at $P$.*

*In particular, for any $U$ open in $X$, the locus on which any regular differential form $\omega \in \Omega(U)$ vanishes is closed in $U$.*

We conclude immediately from the second statement that regular differential forms satisfy the same rigidity property as regular functions.

**Lemma 1.8.** *Suppose $U \subseteq V \subseteq X$ are open subsets. If two regular differential forms on $V$ are equal after restriction to $U$, then they are equal on $V$.*

We can thus define a rational differential form just as we did a rational function.

**Definition 1.9.** A **rational differential form** on $X$ is an equivalence class of pairs $\langle U, \omega\rangle$ where $U \subseteq X$ is open, and $\omega$ is a regular differential form on $U$. The equivalence relation is that $\langle U, \omega\rangle \sim \langle V, \omega'\rangle$ if $\omega|_{U\cap V} = \omega'|_{U\cap V}$.

The rational differential forms are clearly a vector space over $K(X)$. We conclude our general discussion of differentials with a description of the rational differential forms on $X$.

**Proposition 1.10.** *The rational differential forms on $X$ have dimension over $K(X)$ equal to $\dim X$, and indeed if $P \in X$ is any point, and $t_1, \ldots, t_n \in \mathcal{O}_{P,X}$ a basis of $\mathfrak{m}_P/\mathfrak{m}_P^2$, where $n = \dim X$, then $dt_1, \ldots, dt_n$ form a basis of the rational differential forms on $X$ over $K(X)$.*

*Proof.* We know from Lemma 1.6 that there exists an open neighborhood $U$ of $P$ on which every $\omega \in \Omega(U)$ can be written uniquely as $\sum_i f_i dt_i$ for $f_i \in \mathcal{O}(U)$, and that in fact the same holds for every $V \subseteq U$. But then the desired statement is clear, since every rational differential form has a representative on some $V \subseteq U$, as does every rational function. □

## 2. Differential forms on curves

Just as with rational functions, if $X$ is a nonsingular curve, and $\omega$ a rational differential form on $X$, we have a notion of order of zeroes or poles of $\omega$ at points on $X$, and we can associate a divisor $D(\omega)$ to $\omega$. We assume throughout this section that $X$ is a nonsingular.

**Definition 2.1.** If $\omega$ is a nonzero rational differential form on $X$, we define the associated divisor $D(\omega)$ on $X$ as follows: for any $P \in X$, let $t$ be a local coordinate, and write $\omega = f dt$ for some $f \in K(X)^*$. Then the coefficient of $[P]$ in $D(\omega)$ is $\nu_P(f)$.

**Proposition 2.2.** *The divisor $D(\omega)$ is a well-defined divisor on $X$. It has nonnegative coefficient at $P$ if and only if $\omega$ is regular in a neighborhood of $P$, and strictly positive coefficient at $P$ if and only if $\omega$ vanishes at $P$.*

*Proof.* By Proposition 1.10 we have that $\omega = f dt$ for a uniquely determined $f$, and by Lemma 1.8 if $t, t'$ are two local coordinates, then $dt$ and $dt'$ can each be written as regular multiples of one another, so we must have $dt' = g dt$ for some $g$ regular and nonvanishing at $P$, and we find that $D(\omega)$ is well-defined at each point.

If $D(\omega)$ has nonnegative coefficient at $P$, then $f$ is regular at (and therefore in a neighborhood of) $P$, so $\omega$ is as well. Conversely, if $\omega$ is regular at $P$, we know from Lemma 1.6 that $\omega$ can be written as $f dt$ for $f$ regular on a neighborhood of $P$, so $D(\omega)$ has nonnegative coefficient at $P$. Similarly, since $dt$ spans $\mathfrak{m}_P/\mathfrak{m}_P^2$, it is clear that $\omega$ vanishes at $P$ if and only if $f$ does, if and only if the coefficient of $D(\omega)$ at $P$ is positive.

Finally, we see that $D(\omega)$ is indeed a divisor, because it can have nonzero coefficient at only finitely many points: the points at which $\omega$ is not regular, and the points at which $\omega$ vanishes. □

We can thus define particular spaces of rational differential forms subject to vanishing conditions:

**Definition 2.3.** Denote by $\Omega(D)$ the $k$-vector space of rational differential forms $\omega$ on $X$ such that $\omega = 0$ or $D(\omega) + D \geqslant 0$.

We have the following easy consequence of the definition of $D(\omega)$:

**Proposition 2.4.** *Given any $f \in K(X)^*$ and nonzero rational differential form $\omega$, we have*

$$D(f\omega) = D(f) + D(\omega).$$

Two key facts, both following from Proposition 1.10, are the following:

**Corollary 2.5.** *Given any two nonzero rational differential forms $\omega, \omega'$ on $X$, we have that $D(\omega)$ and $D(\omega')$ are linearly equivalent.*

*Proof.* Given Proposition 2.4, this follows immediately from Proposition 1.10 which implies that $\omega' = f\omega$ for some $f \in K(X)^*$. □

**Corollary 2.6.** *For any divisor $D$, the space $\Omega(D)$ is finite-dimensional over $k$.*

*Proof.* Let $K = D(\omega)$ for some nonzero rational differential form $\omega$ on $X$. By Proposition 1.10, every other such form $\omega'$ can be written uniquely as $f\omega$ for some $f \in K(X)^*$, and by Proposition 2.4 we have $D(\omega') = D(f) + K$. It follows immediately that $\Omega(D)$ is isomorphic to $\mathcal{L}(K + D)$ via the map $\omega' \mapsto f$. □

In particular, we are now able to make the following fundamental definition.

**Definition 2.7.** If $X$ is a nonsingular projective curve, we define the **genus** of $X$ to be the dimension over $k$ of $\Omega(X)$.

**Example 2.8.** If $X = \mathbb{P}^1$, let $t$ be a coordinate on $\mathbb{A}^1 \subseteq \mathbb{P}^1$. Then a differential form regular on $\mathbb{A}^1$ is of the form $fdt$ for some $f \in k[t]$, but one checks easily that $dt$ has a pole of order 2 at $\infty$, and therefore no matter what $f$ is, the form $fdt$ cannot be regular at $\infty$. Thus $\mathbb{P}^1$ has genus 0. In fact, we will see later that up to isomorphism, $\mathbb{P}^1$ is the only nonsingular projective curve of genus 0.

## 3. Differential forms and ramification

We next want to study the structure of ramification of morphisms.

**Definition 3.1.** A nonconstant morphism $\varphi : X \to Y$ of curves is **separable** if the induced field extension $K(X)/K(Y)$ is separable. Otherwise, we say $\varphi$ is **inseparable**.

In particular, if $k$ has characteristic 0, every (nonconstant) morphism is separable. We aim to prove the following theorem relating ramification to separability.

**Theorem 3.2.** *Let $\varphi : X \to Y$ be a nonconstant morphism of nonsingular curves. Then $\varphi$ is inseparable if and only if every $P \in X$ is a ramification point of $f$, if and only if infinitely many points of $X$ are ramifications points of $f$.*

In order to prove the theorem, we have to investigate the behavior of differential forms under morphisms. Since a morphism $\varphi : X \to Y$ induces linear maps $T^*_{\varphi(P)} \to T^*_P$ for every $P \in X$, it is clear that given a differential form $\omega$ on $V$, we obtain from $\varphi$ a differential form $\varphi^*(\omega)$ on $\varphi^{-1}(V)$. Moreover, we see that

$$\varphi^* \sum_i g_i df_i = \sum_i \varphi^* g_i d\varphi^* f_i,$$

so if $\omega$ is regular, then $\varphi^*\omega$ is likewise regular. We will be interested in the behavior of pullback of differential forms for morphisms of nonsingular curves. An important preliminary definition is:

**Definition 3.3.** A nonconstant morphism $\varphi : X \to Y$ of curves is **wildly ramified** at $P$ if $\operatorname{char} k = p > 0$ and $p | e_P$. If $P$ is a ramification point at which $\varphi$ is not wildly ramified, we say $\varphi$ is **tamely ramified** at $P$. We say $\varphi$ is **tamely ramified** if every $P$ is either unramified or tamely ramified.

We can then state the relationship between ramification and pullback of differential forms as follows.

**Proposition 3.4.** *Given a nonconstant morphism $\varphi : X \to Y$ of nonsingular curves, and $P \in X$, and $t$ a local coordinate at $\varphi(P)$, then $P$ is a ramification point of $\varphi$ if and only if $\varphi^* dt$ vanishes at $P$.*

*More precisely, if $\varphi^* dt \neq 0$, then it vanishes to order at least $e_P - 1$ at $P$, and $\varphi$ is wildly ramified at $P$ if and only if we either have strict inequality, or $\varphi^* dt = 0$.*

*Proof.* By definition, we have $\varphi^* dt = d\varphi^* t$. On the other hand, $\varphi^* t = g s^{e_P}$, where $s$ is a local coordinate at $P$, and $g$ is a nonvanishing regular function on a neighborhood of $P$. Thus,

$$\varphi^* dt = d(g s^{e_P}) = s^{e_P} dg + e_P s^{e_P - 1} g ds.$$

We see that if this is nonzero, it vanishes to order at least $e_P - 1$, as claimed. Morever, $s^{e_P} dg$ vanishes to order at least $e_P$, and $s^{e_P - 1} g ds$ vanishes to order exactly $e_P - 1$, so we conclude that $\varphi^* dt$ is either identically zero or vanishes to order strictly greater than $e_P - 1$ if and only if $e_P = 0$ in $k$, which is exactly the case of wild ramification. ∎

The basic behavior of inseparable extensions in the case of transcendence degree 1 is the following:

*Exercise* 3.5. Suppose $L/K$ is an algebraic extension of fields of characteristic $p > 0$, and $f \in L$ has a minimal polynomial $h(t) \in K[t]$ such that each coefficient of $h$ is a $p$th power in $K$, then $f = g^p$ for some $g \in L$.

**Proposition 3.6.** *Given $f \in K(X) \smallsetminus k$, we have $df = 0$ if and only if $\operatorname{char} k = p > 0$ and $f = g^p$ for some $g \in K(X)$.*

*Proof.* If $f = g^p$, we have $df = pg^{p-1}dg = 0$ in characteristic $p$, by Exercise 1.5. For the converse, let $t$ be a local coordinate at any point of $X$, so that we know from Proposition 1.10 that $dt$ is a basis over $K(X)$ for the rational differential forms on $X$; in particular, $gdt = 0$ on any open subset if and only if $g = 0$. Now, $k(t)$ has transcendence degree 1, so $K(X)$ is algebraic over $k(t)$; in particular, $f$ satisfies a polynomial relation $h(f) = 0$ for some $h \in k(t)[z]$. We may assume that $h$ is the minimal polynomial of $f$, and in particular irreducible. Since $f \notin k$, and $k$ is algebraically closed, we have that at least one coefficient of $h$ is not in $k$. Clearing denominators if necessary, we may assume that the coefficients of $h$ are in $k[t]$, with no common factors. Writing $h(z) = \sum_i h_i z^i$ and applying Exercise 1.5 (considering $h$ as a polynomial in $t$ and $z$) and the hypothesis that $df = 0$, we have

$$0 = d(h(f)) = \frac{dh}{dz}(f)df + \frac{dh}{dt}(f)dt = \sum_i \frac{dh_i}{dt} f^i dt,$$

so $\sum_i \frac{dh_i}{dt} f^i = 0$. We conclude that $\sum_i \frac{dh_i}{dt} z^i$ is a polynomial having $f$ as a root, but it has at most the same degree as the minimal polynomial for $f$, and the degree in $t$ of the coefficients is strictly smaller, which by uniqueness of the minimal polynomial is not possible unless $\frac{dh_i}{dt} = 0$ for all $i$. We conclude that each $h_i$ has nonzero coefficients only for powers of $t$ which are multiples of $p$; since $k$ is algebraically closed, each $h_i$ is a $p$th power. Thus, we conclude $f = g^p$ for some $g \in K(X)$ by Exercise 3.5. □

*Exercise* 3.7. Suppose $k$ has characteristic $p > 0$, and $K$ is finitely generated of transcendence degree 1 over $k$.

(a) Prove that $K^p$ is a subfield of $K$, and $K$ has degree $p$ over $K^p$.

(b) Prove that if $L$ is a subfield of $K$, also of transcendence degree 1 over $k$, then $K/L$ is inseparable if and only if $L \subseteq K^p$.

*Remark* 3.8. The geometric content of Exercise 3.7 is that a nonconstant morphism $X \to Y$ of nonsingular curves is inseparable if and only if it factors through a certain Frobenius map $X \to X^{(p)}$.

We can now prove the theorem.

*Proof of Theorem 3.2.* It follows from and Exercise 3.7 that if $\varphi$ is inseparable, then for any $f \in K(Y)$, there exists $g \in K(Y)$ such that $\varphi^* f = g^p$. Given $P \in X$, applying this to the case that $f$ is a local coordinate at $\varphi(P)$, we see that $\varphi^* f$ must vanish to order a multiple of $p$ at $P$, and therefore $f$ is a ramification point. We thus wish to show that if $\varphi$ is separable, it is ramified at only finitely many points of $X$. Let $t$ be a local coordinate at some point $Q \in Y$; then since $t$ has valuation 1 at $Q$, we have that $t$ is not a $p$th power in $K(Y)$. It follows from separability of $K(X)$ over $K(Y)$ that $\varphi^* t$ is not a $p$th power in $K(X)$, and thus $\varphi^* dt = d\varphi^* t \neq 0$ by Proposition 3.6. Now, we know that $t - t(Q')$ is a local coordinate at $Q'$ for $Q'$ in some open neighborhood $V$ of $Q$, so by Proposition 3.4, on $\varphi^{-1}(V)$ the ramification of $\varphi$ is determined by the vanishing of $\varphi^* d(t - t(Q')) = \varphi^* dt$. But because $\varphi^* dt \neq 0$, it vanishes at only finitely many points of $\varphi^{-1}(V)$, and we conclude that $\varphi$ can only be ramified at those points or in $X \smallsetminus \varphi^{-1}(V)$, which is also a finite set. □

Since we cannot have inseparable extensions in characteristic 0, we conclude the following immediately from Theorem 3.2.

**Corollary 3.9.** *If* $\operatorname{char} k = 0$*, and* $\varphi : X \to Y$ *is a nonconstant morphism of nonsingular curves, then* $\varphi$ *has only finitely many ramification points.*