PROJECTIVE VARIETIES

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We now move on to studying projective varieties. Unlike Hartshorne's approach, we will treat them as an example of the more general abstract varieties we have defined.

1. Projective space

We will first construct projective space $\mathbb{P}^n_k$ as a prevariety, and verify that it is a variety. As a set, we define

$$\mathbb{P}^n_k = \{(a_0, \dots, a_n) : a_i \neq 0 \text{ for at least one } i\} / \sim,$$

where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if there exists $\lambda \in k^*$ such that $(a_0, \dots, a_n) = \lambda(b_0, \dots, b_n)$.

To define the topology, we recall that a polynomial $F \in k[X_0, \dots, X_n]$ is **homogeneous** if all its monomials have the same total degree. We then observe that if $F \in k[X_0, \dots, X_n]$ is homogeneous, although F does not define a function on $\mathbb{P}^n_k$ because of the equivalence relation, we have

$$F(\lambda a_0, \dots, \lambda a_n) = \lambda^d F(a_0, \dots, a_n),$$

where d is the degree of F. Thus, F has a well-defined zero set in $\mathbb{P}^n_k$, which we denote by $Z(F) \subseteq \mathbb{P}^n_k$. More generally, if $S \subseteq k[X_0, \dots, X_n]$ consists entirely of homogeneous polynomials (not necessarily all of the same degree), we can define

$$Z(S) = \bigcap_{F \in S} Z(F) \subseteq \mathbb{P}^n_k.$$

Definition 1.1. A subset of $\mathbb{P}^n_k$ is **algebraic** if it is of the form $Z(S)$ for some collection S of homogeneous polynomials.

Proposition 1.2. *We have:*

(i) $Z(1) = \emptyset$ *and* $Z(0) = \mathbb{P}^n_k$*;*
(ii) $\bigcap_{i \in I} Z(S_i) = Z(\bigcup_{i \in I} S_i)$*;*
(iii) $Z(S_1) \cup Z(S_2) = Z(S_1 S_2)$*.*

The proof is the same as in the affine case, or can even be seen to follow from the affine case, considering zero sets inside of $\mathbb{A}^{n+1}_k$.

This means we can define the topology on $\mathbb{P}^n_k$ to have closed sets consisting precisely of algebraic subsets.

It remains to give $\mathbb{P}^n_k$ an atlas. For this, choose an index i, and observe that if we set $U_i = \mathbb{P}^n_k \smallsetminus Z(X_i)$, then every element of U_i has a unique representative $(a_0, \dots, a_n)$ with $a_i = 1$. We thus define $\varphi_i : \mathbb{A}^n_k \to U_i$ by $\varphi_i(b_1, \dots, b_n) = (b_1, \dots, b_i, 1, b_{i+1}, \dots, b_n)$. We claim:

Proposition 1.3. *The map φ_i is a homeomorphism. Moreover, for all $i \neq j$, the map*

$$\varphi_{i,j} : \varphi_i^{-1}(U_j) \xrightarrow{\varphi_i} U_i \cap U_j \xrightarrow{\varphi_j^{-1}} \varphi_j^{-1}(U_i)$$

is a morphism.

Equivalently, $\{U_i, \varphi_i\}$ define an atlas for $\mathbb{P}^n_k$ as a prevariety.

Proof. The map φ_i is visibly bijective, so to see it is a homeomorphism, we need to check that closed subsets coincide on both sides. Given a polynomial $f \in k[x_1, \ldots, x_n]$ of degree d, let $h_i(f) \in k[X_0, \ldots, X_n]$ be obtained as follows: if $f = \sum_{(j_1,\ldots,j_n)} a_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}$, set

$$h_i(f) = \sum_{(j_1,\ldots,j_n)} a_{j_1,\ldots,j_n} X_0^{j_1} \cdots X_{i-1}^{j_i} X_i^{d-\sum_\ell j_\ell} X_{i+1}^{j_{i+1}} \cdots X_n^{j_n},$$

so that $h_i(f)$ is homogeneous of degree d. On the other hand, if $F \in k[X_0, \ldots, X_n]$ is homogeneous of degree d, then let $d_i(F) \in k[x_1, \ldots, x_n]$ be obtained as follows: if $F = \sum_{(j_0,\ldots,j_n):\sum_\ell j_\ell = d} a_{j_0,\ldots,j_n} X_0^{j_0} \cdots X_n^{j_n}$, set

$$d_i(F) = \sum_{(j_0,\ldots,j_n):\sum_\ell j_\ell = d} a_{j_0,\ldots,j_n} x_1^{j_0} \cdots x_i^{j_{i-1}} x_{i+1}^{j_{i+1}} \cdots x_n^{j_n}.$$

Then we see that $\varphi_i(Z(f)) = Z(h_i(f)) \cap U_i$ for any $f \in k[x_1, \ldots, x_n]$, and $\varphi_i^{-1}(Z(F)) = Z(d_i(F))$ for any homogeneous $F \in k[X_0, \ldots, X_n]$. Since the topologies on $\mathbb{A}_k^n$ and U_i are obtained by intersecting sets of the form $Z(f)$ and $Z(F) \cap U_i$ respectively, we see that φ_i is a homeomorphism.

Now, for $i \neq j$, we verify that the transition map $\varphi_{i,j} : \varphi_i^{-1}(U_j) \to \varphi_j^{-1}(U_i)$ is a morphism. But from the definitions, we see that it is given by

$$\begin{aligned}(a_1, \ldots, a_n) &\mapsto (a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_n) \\ &\mapsto \begin{cases} (a_1/a_j, \ldots, a_i/a_j, 1/a_j, a_{i+1}/a_j, \ldots, a_{j-1}/a_j, a_{j+1}/a_j, \ldots, a_n/a_j) : & i < j \\ (a_1/a_{j+1}, \ldots, a_j/a_{j+1}, a_{j+2}/a_{j+1}, \ldots, a_i/a_{j+1}, 1/a_{j+1}, a_{i+1}/a_{j+1}, \ldots, a_n/a_{j+1}) : & i > j. \end{cases}\end{aligned}$$

But we also have

$$\varphi_i^{-1}(U_j) = \begin{cases} \{(a_1, \ldots, a_n) : a_j \neq 0\} : & i < j \\ \{(a_1, \ldots, a_n) : a_{j+1} \neq 0\} : & i > j, \end{cases}$$

so we see that $\varphi_{i,j}$ is a morphism, as desired. □

Thus, we see that $\mathbb{P}_k^n$ is a prevariety. In fact, it is a variety, as we will see shortly. However, we first make some basic definitions, and study morphisms to projective varieties.

Definition 1.4. A **projective variety** is a closed subprevariety of $\mathbb{P}_k^n$ for some n. A **quasiprojective variety** is a subprevariety of $\mathbb{P}_k^n$ for some n.

2. Morphisms to projective varieties

We will begin by studying regular functions on quasiprojective varieties, giving a rational function description analogous to the quasiaffine case. As we have mentioned, a homogeneous polynomial F of degree d doesn't define a function on $\mathbb{P}_k^n$ (we instead refer to it as a **form**), but a quotient of two of them does – at least, where the denominator is nonvanishing. This leads to the following description.

Proposition 2.1. *Let $X \subseteq \mathbb{P}_k^n$ be a quasiprojective variety, $U \subseteq X$ open, and $f : U \to k$ a function. Then f is regular if and only if for all $P \in U$, there exists V an open neighborhood of P and $F, G \in k[X_0, \ldots, X_n]$ homogeneous of equal degree such that $V \cap Z(G) = \emptyset$ and $f = F/G$ on V.*

Proof. Clearly, since F and G have the same degree, F/G defines a function k. It is straightforward to check from our atlas for $\mathbb{P}_k^n$ that this function is regular: using the notation from the proof of Proposition 1.3, if $P \in U_i$, then F/G agrees with $d_i(F)/d_i(G)$ under φ_i, and is defined on $\varphi_i^{-1}(V)$.

Conversely, if f is regular and $P \in U_i$, then $f \circ \varphi_i : U_i \cap U \to k$ is regular, so there is some $V_i \subseteq X_i$ a neighborhood of $\varphi_i^{-1}(P)$ on which $f \circ \varphi_i = g/h$ for some $g, h \in k[x_1, \ldots, x_n]$, with $V_i \cap Z(h) = \emptyset$. Then on $\varphi_i(V_i) \subseteq U_i$, we have $f = X_i^d h_i(g)/h_i(h)$, where $d = \deg h - \deg g$. □

Next, recall that if $Y \subseteq \mathbb{A}^n_k$ is affine, then for any prevariety X, morphisms $X \to Y$ are the same as n-tuples of regular functions on X such that the induced map $X \to \mathbb{A}^n_k$ has image contained in Y. A similar, but slightly more complicated statement holds in the case that $Y \subseteq \mathbb{P}^n_k$ is projective.

In this case, if we have an $(n+1)$-tuple $(f_0, \dots, f_n)$ of regular functions on X, and if further there is no point of X at which all the f_i are zero, then we get a map $X \to \mathbb{P}^n_k$. However, not every morphism can be globally described by such an $(n+1)$-tuple. The correct statement is as follows:

Proposition 2.2. *Given a map $\varphi : X \to Y$, with X any prevariety and $Y \subseteq \mathbb{P}^n$ a projective variety., φ is a morphism if and only if it is described locally by $(n+1)$-tuples of regular functions which do not all vanish simultaneously.*

We first prove the following general lemma:

Lemma 2.3. *Let X be a prevariety, U an open subset, and $f : U \to k$ regular. Then f is a unit in $\mathcal{O}(U)$ if and only if $f(P) \neq 0$ for all $P \in U$.*

Proof. This is clear from the definitions in the case that X is affine, and the general prevariety case reduces to the affine case by the definition of regularity for prevarieties. □

Proof of Proposition 2.2. First suppose φ is a morphism, and let $V_j = Y \smallsetminus Z(x_j)$ for $j = 0, \dots, n$. Then on each V_j we have the regular functions induced by $\frac{x_i}{x_j}$ on $\mathbb{P}^n$ for $i = 0, \dots, n$, which gives us an $(n+1)$-tuple of regular functions on $\varphi^{-1}(V_j)$ by composing with φ. Noting that in $\mathbb{P}^n \smallsetminus Z(x_j)$ the point $(c_0, \dots, c_n)$ is also represented by $(\frac{c_0}{c_j}, \dots, \frac{c_n}{c_j})$, and that $\frac{x_i}{x_i} = 1$ vanishes nowhere on $\varphi^{-1}(V_j)$, we see that φ is represented by this $(n+1)$-tuple on $\varphi^{-1}(V_j)$. Letting j vary, we find that φ is everywhere locally represented by $(n+1)$-tuples of regular functions which do not all vanish simultaneously.

Conversely, suppose that we have some open cover U_i of X such that on each U_i, we can express φ as an $(n+1)$-tuple of regular functions which do not all vanish simultaneously. By refining the U_i, we may assume they are affine. Since being a morphism is a local condition on X, it is enough to see that if we consider each U_i as a prevariety, then φ induces a morphism $U_i \to Y$. Moreover, since $U_i \to Y$ is a morphism if and only if the composed map $U_i \to \mathbb{P}^n$ is a morphism, it suffices to treat the case $Y = \mathbb{P}^n$. Thus, we have reduced to the case that X is affine, $Y = \mathbb{P}^n$, and φ is given globally by an $(n+1)$-tuple of regular functions $f_0, \dots, f_n$ on X. But if for $j = 0, \dots, n$ we write $U_j := X \smallsetminus Z(f_j)$, then by hypothesis the U_j cover X, and each $U_j = \varphi^{-1}(\mathbb{P}^n \smallsetminus Z(x_j))$. Taking coordinates $\frac{x_i}{x_j}$ (with $i \neq j$) on $\mathbb{P}^n \smallsetminus Z(x_j)$, the map $U_j \to \mathbb{P}^n \smallsetminus Z(x_j)$ is then given by the n-tuple of functions $\frac{f_i}{f_j}$ for $i \neq j$, which are regular by Lemma 2.3. Then our map $U_j \to \mathbb{P}^n \smallsetminus Z(x_j)$ is a morphism for every j and because being a morphism is local on the target, we conclude that φ is a morphism. □

In the case of maps between quasiprojective varieties, we obtain the following.

Corollary 2.4. *Let $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_k$ be quasiprojective varieties. Given $F_0, \dots, F_m \in k[X_0, \dots, X_n]$ homogeneous of degree d, suppose that $Z(F_0) \cap \dots \cap Z(F_m) \cap X = \emptyset$, and the induced map $X \to \mathbb{P}^m_k$ has image contained in Y. Then the resulting map $X \to Y$ is a morphism.*

More generally, a map $X \to Y$ which can be expressed as above locally on X is a morphism.

Proof. First, we note that there is in fact an induced map $X \to \mathbb{P}^m_k$, since scaling the X_i by λ scales each F_i by λ^d. Now, given $P \in X$, by hypotheses there is some i such that $F_i(P) \neq 0$, so $U = X \smallsetminus Z(F_i)$ is an open neighborhood of P on which we can represent our map by $(F_0/F_i, \dots, F_m/F_i)$. We conclude from Propositions 2.1 that the map to Y is given locally on X by $(m+1)$-tuples of regular functions, and then Proposition 2.2 implies that we have a morphism, as desired. □

Example 2.5. Any linear change of coordinates is an automorphism $\mathbb{P}^n_k \xrightarrow{\sim} \mathbb{P}^n_k$. Indeed, it is given by an $(n+1)$-tuple of linearly independent homogeneous linear polynomials $(F_0, \ldots, F_n)$, so the common zero set is the origin in k^{n+1}, and empty in $\mathbb{P}^n_k$. By Corollary 2.4, it is a morphism. But the inverse map is of the same form, so we conclude that we have an automorphism, as desired.

In particular, we see that if $H \subseteq \mathbb{P}^n_k$ is a hyperplane, then $\mathbb{P}^n_k \smallsetminus H \cong \mathbb{A}^n_k$, since there is an automorphism mapping $Z(x_0)$ to H.

We can now conclude our previously promised result.

Corollary 2.6. *Any quasiprojective variety is a variety.*

Proof. It is enough to show that $\mathbb{P}^n_k$ is a variety for any n. By our criterion for varieties, it is enough to show that any two points of $\mathbb{P}^n_k$ are contained in an open subset which is a variety. But we can always find a hyperplane H which does not contain any two given points, and then $\mathbb{P}^n_k \smallsetminus H \cong \mathbb{A}^n_k$ by Example 2.5, so we conclude the desired statement. □

Example 2.7. What if we take Example 2.5, but use fewer linear forms? Say we have $F_0, \ldots, F_m$ linearly independent homogeneous linear polynomials, for some $m < n$. Then $(F_0, \ldots, F_m)$ defines a rational map $\mathbb{P}^n_k \dashrightarrow \mathbb{P}^m_k$, which is a morphism away from the set $Z = Z(F_0) \cap \cdots \cap Z(F_m)$. We see that Z is a linear subspace of $\mathbb{P}^n_k$, of dimension $n - m - 1$. This rational map is called **linear projection** from Z. It can be described geometrically as follows: choose another linear subspace $Z' \subseteq \mathbb{P}^n_k$, of dimension m, with $Z' \cap Z = \emptyset$. Then given a suitable choice of coordinates on Z', our map can be expressed as follows: it is the morphism $\mathbb{P}^n_k \smallsetminus Z \to Z'$ which sends a point P to the point $Q \in Z'$ which is the unique intersection point of Z' with the linear span of P and Z. (Notice that the linear span of P and Z is a linear space of dimension $n - m$, so meets Z' in one point)

Example 2.8. Consider $X = Z(XY - Z^2) \subseteq \mathbb{P}^2_k$. We claim that $X \cong \mathbb{P}^1_k$. We can construct a morphism $X \to \mathbb{P}^1_k$ by linear projections as follows: away from $(0,1,0)$, we have $(X,Y,Z) \mapsto (X,Z)$, and away from $(1,0,0)$, we have $(X,Y,Z) \mapsto (Z,Y)$. On X, away from both $(0,1,0)$ and $(1,0,0)$ we have $X/Z = Z/Y$, so we see that these two maps together yield a morphism $X \to \mathbb{P}^1_k$. The inverse is given by $(S,T) \mapsto (S^2, T^2, ST)$, so we get the claimed isomorphism.

3. Projective varieties as complete varieties

The value of working with projective varieties is that they are, in an intuitive sense, compact. In order to make this precise, we have to address the problem that in the Zariski topology, every variety is compact. As with the Hausdorff condition, the solution is provided by finding an equivalent definition of compactness for usual topological spaces which involves products.

Exercise 3.1. A topological space X is compact if and only if for every topological space Y, the projection map $X \times Y \to Y$ is a closed map.

Hint: given a collection of open subsets $\{U_i\}$ of X, if no finite number of them cover X, define Y by adding a single point ω to X, with the topology whose open subsets consist of: any subset of X; and any set containing a set of the form $\{\omega\} \cup (X \smallsetminus U)$, where U is a finite union of sets U_i. Show that the U_i cannot cover X.

Motivated by this, we define:

Definition 3.2. A variety X is **complete** if for all varieties Y, the projection morphism $X \times Y \to Y$ is a closed map.

Remark 3.3. One can apply the definition of completeness to prevarieties as well, but it is traditional to reserve the term "complete" for varieties. This is related to the French tradition that compactness should incorporate the Hausdorff condition as well.

Exercise 3.4. One could define a prevariety X to be **universally closed** if for all prevarieties Y, the projection map $X \times Y \to Y$ is closed. Thus, if X is universally closed and is also a variety, then X is complete. Show that conversely, if X is a complete variety, then X is universally closed. Show the stronger statement that a prevariety X is universally closed if and only if $X \times \mathbb{A}^n_k \to \mathbb{A}^n_k$ is closed for all n.

Part of the power of completeness comes from its relationship to closed morphisms.

Proposition 3.5. *If X is complete, any closed subvariety of X is complete.*
If we also have Y an arbitrary variety, and $\varphi : X \to Y$ a morphism, then φ is closed.

Proof. The first assertion is immediate from the definition, since if Z is closed in X, we have $Z \times Y$ closed in $X \times Y$ for any Y.

Given the first assertion, the second assertion reduces to proving that $\varphi(X)$ is closed in Y. Let $\Gamma = \{(x, \varphi(x) : x \in X\} \subseteq X \times Y$ be the graph of φ. We can express Γ as the preimage of the diagonal $\Delta(Y) \subseteq Y \times Y$ under the morphism $\varphi \times \mathrm{id} : X \times Y \to Y \times Y$. Since Y is a variety, $\Delta(Y)$ is closed, so Γ is closed. But $\varphi(X)$ is precisely the image of Γ under the projection $X \times Y \to Y$, so we conclude from the completeness hypothesis on X that $\varphi(X)$ is closed. □

Remark 3.6. This is a natural property for complete varieties to have, since a continuous map from a compact topological space to a Hausdorff space is closed. In fact, this property characterizes complete varieties – Nagata proved that every variety can be realized as an open subset of a complete variety, so in particular if X is not complete, the inclusion as an open subset of a complete variety is not a closed mapping. However, the proof of Nagata's theorem is beyond the scope of this course.

We now observe that complete varieties behave very differently from (quasi)affine varieties:

Proposition 3.7. *Let X be a complete variety. Then every regular function on X is constant; that is, $\mathscr{O}(X) = k$.*

Proof. A regular function f on X yields a morphism $X \to \mathbb{A}^1_k$. We can compose with the inclusion $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ to obtain a morphism $\varphi : X \to \mathbb{P}^1_k$. Now, X is complete and $\mathbb{P}^1_k$ is a variety, so by Proposition 3.5, we have $\varphi(X)$ closed in $\mathbb{P}^1_k$. On the other hand, X is irreducible, and the continuous image of an irreducible space is irreducible, so $\varphi(X)$ is closed and irreducible in $\mathbb{P}^1_k$. The only closed irreducible subsets of $\mathbb{P}^1_k$ are all of $\mathbb{P}^1_k$ or points, and since $\varphi(X) \subseteq \mathbb{A}^1_k$, we cannot have $\varphi(X) = \mathbb{P}^1_k$, so $\varphi(X)$ is a point, and the original regular function was constant, as desired. □

As a consequence, we cannot imbed any positive-dimensional complete variety into affine space. In fact, we have the following.

Corollary 3.8. *If X is complete, and Y is affine, then any morphism $X \to Y$ is constant.*

Proof. Imbed $Y \subseteq \mathbb{A}^n_k$, a morphism $X \to Y$ induces a morphism $X \to \mathbb{A}^n_k$, which we know is determined by an n-tuple of regular functions. But these are all constant by Proposition 3.7, so the morphism is constant, as claimed. □

The point of this discussion, of course, is that projective varieties are complete. To see this, it is enough by Proposition 3.5 to prove that $\mathbb{P}^n_k$ is complete for any n. One can prove directly, but we won't. We will however describe a key piece of a more conceptual proof, and sketch the remainder of the proof.