NONSINGULAR CURVES

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The primary goal of this note is to prove that every abstract nonsingular curve can be realized as an open subset of a (unique) nonsingular projective curve. Note that this encapsulates two facts in one: that every nonsingular abstract curve is quasiprojective, and that it can be "compactified" into a projective curve without introducing singularities.

We start from the definitions and state the necessary background algebra.

1. Curves, regular functions, and morphisms

Our discussion of the abstract definition of a variety allows us to work transparently with abstract curves.

Definition 1.1. A **curve** is a variety of dimension 1.

We first show the following.

Lemma 1.2. *Let P be a nonsingular point of a curve C. Then there exists an open neighborhood U of P and a regular function t on U such that for all $V \subseteq U$ open containing P, and $f \in \mathscr{O}(V)$, we have $f(P) = 0$ if and only if t divides f in $\mathscr{O}(V)$.*

Consequently, for all $f \in \mathscr{O}(V \smallsetminus \{P\})$, we can write $f = t^{\nu} g$, where $\nu \in \mathbb{Z}$, and $g \in \mathscr{O}(V)$ satisfies $g(P) \neq 0$. Moreover, ν is uniquely determined and independent of choice of t, and $f \in \mathscr{O}(V) \subseteq K(C)$ if and only if $\nu \geqslant 0$.

Proof. By definition of nonsingular, the maximal ideal $\mathfrak{m}_P \subseteq \mathscr{O}_{P,C}$ can be generated by $\dim C = 1$ elements, so let t be any generator. Then t is regular on some open neighborhood of P; if t is regular on some U', then t has finitely many zeroes on U'. Let U be the complement of the zeroes of t other than P. Then t vanishes only at P when considered as a regular function on U. We claim that this implies the desired statement: given $V \subseteq U$ and $f \in \mathscr{O}(V)$, certainly if t divides f in $\mathscr{O}(V)$, then $f(P) = 0$, but conversely, if $f(P) = 0$, then $f \in \mathfrak{m}_P$, so t divides f in $\mathscr{O}_{P,C}$. This implies that f/t determines a regular function on some neighborhood of P. But since $t(Q) \neq 0$ for all $Q \in U \smallsetminus \{P\}$, we have that t is a unit in $\mathscr{O}(U \smallsetminus \{P\})$, so f/t is regular on $V \smallsetminus \{P\}$. Thus, f/t is regular on an open cover of V, and hence it is regular on V, and we conclude that t divides f in $\mathscr{O}(V)$, as desired.

For the second assertion, if f is regular on V, then we can inductively divide out by t in $\mathscr{O}_{P,C}$ until we obtain $g(P) \neq 0$; this process must eventually terminate because $\mathscr{O}_{P,C}$ is Noetherian. In this case, we see that $\nu \geqslant 0$. If f is not regular on V, it is still an element of $K(C)$, which is the fraction field of $\mathscr{O}_{P,C}$. Thus, if $f = h_1/h_2$ with $h_1, h_2 \in \mathscr{O}_{P,C}$, then using the regular case on h_1 and h_2 we get that in $\mathscr{O}_{P,C}$, we can write $f = t^{\nu} h_3/h_4$ for some $\nu \in \mathbb{Z}$, and $h_3, h_4 \in \mathscr{O}_{P,C} \smallsetminus \mathfrak{m}_P = \mathscr{O}^*_{P,C}$. Then we can set $g = h_3/h_4$; this is in $\mathscr{O}_{P,C}$, so is regular in a neighborhood of P, but we also have $g = ft^{-\nu}$, so is regular on $V \smallsetminus \{P\}$ as well. This gives the desired assertion.

Now, if we have $f = t^{\nu_1} g_1 = t^{\nu_2} g_2$, with $\nu_1 \geqslant \nu_2$, then $g_2 = t^{\nu_1 - \nu_2} g_1$, and since $g_2 \in \mathscr{O}^*_{P,C}$, but $t \in \mathfrak{m}_P$ and $g_1 \in \mathscr{O}_{P,C}$, we must have $\nu_1 - \nu_2 = 0$, as desired. Now, if $\nu \geqslant 0$ then obviously $f \in \mathscr{O}(V)$. Conversely, if $\nu < 0$, then $ft^{-\nu} = g_1$, and since $g_1 \notin \mathfrak{m}_P$, we conclude that we cannot have $f \in \mathscr{O}_{P,C}$. Finally, if we have two choices t_1, t_2, we claim that $t_1 = t_2 g$ for some $g \in \mathscr{O}^*_{P,C}$; it then follows immediately that ν is independent of the choice of t. We have $t_1 = t_2^{\nu_1} g_1$ for some

$g_1 \in \mathscr{O}^*_{P,C}$, and also $t_2 = t_1^{\nu_2} g_2$ for some $g_2 \in \mathscr{O}^*_{P,C}$, with both ν_1, ν_2 positive since t_1 and t_2 are regular and vanishing at P. Then $t_1 = t_1^{\nu_1\nu_2} g_2^{\nu_2} g_1$, so $t_1^{\nu_1\nu_2 - 1} \in \mathscr{O}^*_{P,C}$, and we must have $\nu_1\nu_2 = 1$, and hence $\nu_1 = \nu_2 = 1$, as desired. □

Remark 1.3. The argument can be expressed also in terms of discrete valuation rings – if P is a nonsingular point of a curve C, then $\mathscr{O}_{P,C}$ is a DVR.

In the second part of the lemma, we intuitively think of ν as being the order of the zero (if positive) or pole (if negative) of f at P, so we make the following definition:

Definition 1.4. The ν of Lemma 1.2 is the **order** of f at P, denoted $\operatorname{ord}_P(f)$.

A basic result on morphisms from curves to projective varieties is the following.

Theorem 1.5. *If C is a curve, and $P \in C$ a nonsingular point, and Y a projective variety, then every morphism $C \smallsetminus \{P\} \to Y$ extends uniquely to a morphism $C \to Y$.*

Remark 1.6. We saw in the homework that the uniqueness is satisfied for Y any variety, as a consequence of the condition analogous to being Hausdorff which we used to distinguish varieties among prevarieties. Thus, we need only to prove the existence statement for the theorem.

Proof of Theorem 1.5. By Remark 1.6, it suffices to prove the existence of the desired extension. Let $\varphi : C \smallsetminus \{P\} \to Y$ be the given morphism. Let $U \ni P$ be an open subset such that there exists a t as in Lemma 1.2, and such that on $U \smallsetminus \{P\}$, we can represent φ by an $(n+1)$-tuple of regular functions $f_0, \dots, f_n \in \mathscr{O}(U \smallsetminus \{P\})$ which do not simultaneously vanish anywhere on $\mathscr{O}(U \smallsetminus \{P\})$.

By Lemma 1.2, we can write each f_i as $t^{e_i} g_i$, where $e_i = \operatorname{ord}_P(f_i) \in \mathbb{Z}$ and g_i is regular on U, with $g_i(P) \neq 0$. Choose j with e_j minimal; then since $(f_0, \dots, f_n)$ represents φ on $U \smallsetminus \{P\}$, and t is nonvanishing on this subset, scaling simultaneously by t^{-e_j} we find that $(t^{e_0 - e_j} g_0, \dots, t^{e_n - e_j} g_n)$ also represents φ on the same subset. But $e_i \geqslant e_j$ for all i, so these functions are regular on all of U, and $t^{e_j - e_j} g_j$ is non-zero at P, so setting $\varphi(P) = (t^{e_0 - e_j} g_0(P), \dots, t^{e_n - e_j} g_n(P))$ gives an extension of φ to U.

Finally, being a morphism is a local condition, so if we have extended φ to a morphism on U, since it was already a morphism on $C \smallsetminus \{P\}$, we conclude that we have extended φ to a morphism on all of C. □

Corollary 1.7. *If two nonsingular projective curves are birational, then they are isomorphic.*

Proof. We have an isomorphism of open subsets, but by the theorem each map extends to a morphism on the whole curve. □

Remark 1.8. The idea of extending morphisms of nonsingular curves as in the theorem plays an important role in algebraic geometry, more or less replacing the use of convergent sequences in metric topology. Even though we don't know what it means for points to be close to one another over an arbitrary field, if $\varphi : C \smallsetminus \{P\} \to Y$ extends as in the theorem we can think of $\varphi(P)$ as representing $\lim_{Q \to P} \varphi(Q)$. In particular, we can define a notion analogous to compactness using existence of such extensions; in this language, the theorem is saying that projective varieties are compact, because limits always exist. We also see that the defining condition for varieties can be thought of as saying that limits are unique, when they exist (this is certainly a necessary condition for being Hausdorff; for prevarieties, it turns out to be sufficient, as well).

In fact, the existence of "limits" as above turns out to be equivalent to completeness. In particular, one can use Theorem 1.5 to prove that projective varieties are complete. The idea for this direction is as follows: one proves a lemma that if we have a morphism $f : X \to Y$ of prevarieties, and $Q \in Y$ is in the closure of $f(X)$, then there exists a curve C with a nonsingular point P, and a morphism $C \smallsetminus \{P\} \to X$ such that the composition $C \smallsetminus \{P\} \to X \to Y$ extends to a morphism

$C \to Y$ sending P to Q. Now, suppose that X is a variety with the property that every such morphism $C \smallsetminus \{P\} \to X$ extends to a morphism $C \to X$. Then for any variety Y, and $Z \subseteq X \times Y$ closed, suppose Q is in the closure $p_2(Z) \subseteq Y$, where p_2 is the second projection map. Then according to our lemma, there is a morphism $\varphi : C \smallsetminus \{P\} \to Z$ such that the composition with p_2 extends to a morphism $\psi : C \to Y$ mapping P to Q. Then we can extend $p_1 \circ \varphi$ to a morphism $C \to X$, and taking the product with ψ gives a morphism $\overline{\varphi} : C \to X \times Y$, which must have image contained in Z because Z is closed. Also, we must have $p_2 \circ \overline{\varphi} = \psi$, so we conclude that $\overline{\varphi}(P) \in Z$ maps to Q. This implies that $p_2(Z)$ is closed, and X is complete, as desired.

2. Quasiprojectivity

We will now prove the following theorem:

Theorem 2.1. *If C is a nonsingular curve, then C is quasiprojective.*

For the proof, we need one key background statement, which we organize into an exercise.

Exercise 2.2. (a) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that the composition $U \to Y$ is an isomorphism, then $U = X$.

(b) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that $\varphi : U \to Y$ is an isomorphism onto an open subset $V \subseteq Y$, then $\varphi^{-1}(V) = U$.

(c) Give an example to demonstrate that this is false if X is allowed to be an arbitrary prevariety.

Proof of Theorem 2.1. Let U_i be a cover of C by affine open subsets. Then we have $U_i \subseteq \mathbb{A}^{n_i} \subseteq \mathbb{P}^{n_i}$, so we take Y_i to be the closure of U_i in $\mathbb{P}^{n_i}$. Thus Y_i is projective, and U_i is isomorphic to an open subset of Y_i. By Theorem 1.5, we obtain unique extensions $\varphi_i : C \to Y_i$ for each i (note that each U_i may omit more than one point of C, but we can apply the theorem inductively to extend over each one). These extensions may not be isomorphisms onto their images, because we have little control over what happens when we take the closure of U_i. The trick is to take the product over all i; we then have an induced morphism $\varphi : C \to \prod_i Y_i \subseteq \prod_i \mathbb{P}^{n_i}$. Let $Y \subseteq \prod_i Y_i$ be the closure of the image of C. We will show that C is isomorphic to an open subset of Y. This will prove the theorem, because Y is a closed subset of $\prod_i \mathbb{P}^{n_i}$, which is itself projective via the Segre imbedding (see Exercises 2.14, 3.16 of [1]).

Our first task is to show that φ is a homeomorphism onto an open subset of Y. Now, φ is injective, since given $P, Q \in C$, if $P \in U_i$, we claim $\varphi_i(P) \neq \varphi_i(Q)$. If $Q \in U_i$ as well, this follows from the injectivity of φ_i on U_i, but if $Q \notin U_i$, then $\varphi_i(Q) \notin U_i \subseteq Y_i$ by Exercise 2.2 (b), while $\varphi_i(P) \in U_i$, proving the claim, and thus injectivity. It then follows that φ is a homeomorphism onto its image, since φ maps finite sets to finite sets and thus closed subsets of C to closed subsets of $\varphi(C)$. We next observe that φ is dominant onto Y by definition, so Y is irreducible, and we have $K(Y_i) \hookrightarrow K(Y) \hookrightarrow K(C)$. But $C \to Y_i$ is birational, to $K(Y_i) = K(C)$, and we conclude that $K(Y) = K(C)$, so $\varphi : C \to Y$ is birational. In particular, we conclude that $K(Y)$ has transcendence degree 1, so Y is a curve, and also that $\varphi(C)$ contains an open subset of Y. But since Y is a curve, any subset containing a nonempty open subset is open, so $\varphi(C)$ is open, and we have proved that φ induces a homeomorphism from C onto an open subset of Y.

We now want to see that φ is an isomorphism of C onto its image. It suffices to show that the induced maps on local rings are isomorphisms at every point of C, so let $P \in C$, and consider the induced map $\mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,C}$. This is injective since $C \to Y$ is dominant. Choose i with $P \in U_i$. Then the map $\varphi_i : U_i \to Y_i$ is an isomorphism onto its image, so the induced map $\mathscr{O}_{\varphi_i(P),Y_i} \to \mathscr{O}_{P,U_i}$ is an isomorphism. But we can factor φ_i as $U_i \hookrightarrow C \to Y \to Y_i$, where the last morphism is projection onto the ith factor from the product. This means that the map $\mathscr{O}_{\varphi_i(P),Y_i} \to \mathscr{O}_{P,U_i}$ factors through $\mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,C} = \mathscr{O}_{P,U_i}$, so we conclude that the latter must be surjective, and hence as isomorphism, as desired. $\square$

Remark 2.3. In fact, the theorem holds without the nonsingularity hypothesis, but the proof is a bit more involved. One approach is to show that even on a singular curve, one has an affine open cover such that for every open subset, the omitted points are nonsingular. Given that, the above argument goes through unmodified.

Now that we know that every nonsingular curve is quasiprojective, we can consider the question of projectivity. Obviously, not every nonsingular curve is projective. But we now see that every nonsingular curve can be "compactified" as an open subvariety of a projective curve by imbedding in projective space and taking the closure. But this closure will not in general be nonsingular. So we can ask whether every nonsingular curve can be realized as an open subvariety of a *nonsingular* projective curve. For the moment, although it is clear that every curve is birational to a nonsingular curve, and also to a projective curve, it is not even clear that every curve is birational to a nonsingular projective curve. We will prove the stronger assertion, but only after a discussion of normalization.

3. Normality and normalization

We make a brief detour to discuss the notion of normality. Most of the proofs, while not necessarily difficult, are purely algebraic, and we omit them.

Definition 3.1. A variety X is **normal** if it is covered by affine open subvarieties U_i such that each $A(U_i)$ is integrally closed in $K(X)$.

Recall that given an inclusion of integral domains $R \subseteq S$, an element $s \in S$ is *integral* over R if it is a root of a monic polynomial with coefficients in R. We say R is *integrally closed* in S if every element of S which is integral over R is in fact an element of R. The integers are integrally closed in the rational numbers, by Gauss's lemma, motivating the terminology.

Normality is a somewhat subtle condition, but it does have a fairly direct relationship to nonsingularity. Specifically, we have:

Proposition 3.2. *A nonsingular variety is normal. The singular locus of a normal variety has codimension at least* 2*.*

The proof of this is fairly standard algebra, and we omit it. Since any non-empty closed subset of a curve has codimension at most 1, we conclude:

Corollary 3.3. *A normal curve is nonsingular.*

Another basic algebra statement is:

Proposition 3.4. *X is normal if and only if every affine open subset U has $A(U)$ integrally closed in $K(X)$, if and only if $\mathscr{O}_{P,X}$ is integrally closed in $K(X)$ for all points $P \in X$.*

Remark 3.5. Although it is not our intent to give a complete account of normality, we mention some basic facts for the sake of context. First, the converse to Proposition 3.2 holds for hypersurfaces: if their singular locus has codimension at least 2, then they are normal – see Proposition 2 of §III.8 of [2].

Thus, a variety such as the cone $Z(xy - z^2) \subseteq \mathbb{A}^3_k$ is normal, being a surface with a unique singularity at $(0, 0, 0)$. On the other hand, the surface $Z(w^2y - x^2, w^3z - x^3, y^3 - z^2) \subseteq \mathbb{A}^4_k$, which likewise has $(0, 0, 0, 0)$ as its only singularity, is not normal – see Example K (B) of §III.8 of [2].

Finally, although this doesn't characterize normality, an important property of normality is that a normal variety doesn't have multiple "branches" meeting at a point (i.e., it does not look like a node, or any higher-dimensional analogue).

We will next consider *normalization* – the process of replacing a non-normal variety with a normal one. It turns out that this can be done in a canonical way. The affine version of this process is as follows: if an integral domain R is not integrally closed in its field of fractions, we can take the integral closure, which is the set of all elements of the field which are integral over R. More generally, if $R \subseteq S$ is not integrally closed in S, we can take its integral closure in S. It is a basic algebra fact that the integral closure is again a subring, and it is integrally closed in the field of fractions. The proof is related to the fact that f is integral over R if and only if the ring $R[f]$ is a finitely-generated R-module. More difficult is the the theorem that if R is a finitely-generated k-algebra which is an integral domain, and L a finite extension of the fraction field of R, then the integral closure of R in L is still finitely generated over k. Putting these statements together gives normalization in the affine case.

We are now ready to define the normalization. It will be useful to give a slightly more general form than is immediately necessary.

Definition 3.6. If X is a variety, and L is a finite field extension of $K(X)$ (in particular, algebraic over $K(X)$) the **normalization** $\widetilde{X}_L \to X$ of X in L is the variety constructed by taking an affine open cover U_i of X, letting $\widetilde{U}_i$ be the affine variety determined by the integral closure of $A(U_i)$ in L, and gluing according to injections $A(\widetilde{U}_i) \hookrightarrow L$.

The morphism $\widetilde{X}_L \to X$ is the morphism induced by the inclusions $A(U_i) \hookrightarrow A(\widetilde{U}_i)$.

In particular, the **normalization** $\widetilde{X} \to X$ of X is the normalization of X in $K(X)$.

To make sense of the gluing statement, we have the following basic algebra result:

Exercise 3.7. If R is an integral domain with fraction field K, and L an algebraic extension of K, then the integral closure of R in L has fraction field L.

We then have that each $A(\widetilde{U}_i)$ has fraction field L, so the injections into L induce isomorphisms of fraction fields, and hence birational maps $\widetilde{U}_i \dashrightarrow \widetilde{U}_j$ for all i, j. These birational maps in turn correspond to isomorphisms between open subsets, which we use to define the gluing.

Theorem 3.8. *The normalization $\widetilde{X}_L$ of X in L is a normal variety, and independent of the choice of U_i. The morphism $\widetilde{X}_L \to X$ is surjective, and the induced map on function fields is $K(X) \hookrightarrow K(\widetilde{X}_L) = L$.*

If $U \subseteq X$ is open, and $\widetilde{X}_L \to X$ and $\widetilde{U}_L \to U$ the respective normalizations, then $\widetilde{U}_L$ is naturally identified with the preimage of U in $\widetilde{X}_L$.

Note that in particular, the normalization of X yields a birational morphism.

Remark 3.9. We start to see the utility of having a notion of abstract variety: while it is true that the normalization of a projective variety is projective, the proof isn't trivial, and something of a distraction from the basic idea, that we are simply gluing together integral closures.

Example 3.10. Consider the cuspidal curve $C \subseteq \mathbb{A}^2$ given by $y^2 = x^3$. This is a singular curve, so not normal. We have studied the morphism $\mathbb{A}^1 \to C$ given by $t \mapsto (t^2, t^3)$, corresponding to the injective homomorphism $k[x, y]/(y^2 - x^3) \to k[t]$ sending x to t^2 and y to t^3. This homomorphism induces an isomorphism on fraction fields (this follows from the observation t is the image of $\frac{y}{x}$). We see that t is integral over $A(C)$, since it satisfies $z^2 - x$ (and also $z^3 - y$). But $k[t]$ is integrally closed in its fraction field (one may check this directly, or invoke that nonsingularity implies normality), so we conclude that the morphism $\mathbb{A}^1 \to C$ is in fact the normalization of C.

We see that normalization is universal for (dominant) morphisms from normal varieties:

Proposition 3.11. *Suppose $\varphi : Y \to X$ is a dominant morphism, with Y normal. Then φ factors through the normalization map $\widetilde{X} \to X$.*

Proof. Cover X by affine open subsets U_i, and let V_j be a affine open cover of Y such that each V_j is contained in some $\varphi^{-1}(U_i)$. Let $\widetilde{U}_i$ be the preimages of U_i in $\widetilde{X}$. Fix i, j with $V_j \subseteq \varphi^{-1}(U_i)$. Note that the dominance of φ implies that $A(U_i) \subseteq A(V_j)$. Also $A(U_i) \subseteq A(\widetilde{U}_i)$ by definition. We can consider both these inclusions to hold inside $K(Y)$.

We claim that in fact $A(\widetilde{U}_i) \subseteq A(V_j)$ in $K(Y)$. This follows immediately from the definitions; $A(\widetilde{U}_i)$ is the set of elements of $K(X)$ which are integral over $A(U_i)$, and by the hypothesis that V_i is normal implies that $A(V_j)$ contains all elements of $K(Y)$ which are integral over $A(V_j)$. But because $K(X) \subseteq K(Y)$ and $A(U_i) \subseteq A(V_j)$, any element of $K(X)$ integral over $A(U_i)$ is in particular an element of $K(Y)$ integral over $A(V_j)$, so must lie in $A(V_j)$. We conclude that $A(\widetilde{U}_i) \subseteq A(V_j)$, so the morphisms $V_j \to U_i$ factor through $\widetilde{U}_i \to U_i$. Thus, for each V_j we get that $V_j \to X$ factors through $\widetilde{X}$.

But given j, j' the resulting morphisms $V_j \to \widetilde{X}$ and $V_{j'} \to \widetilde{X}$ both induce the same inclusion $K(X) \to K(Y)$ of function fields by construction, so they define the same rational maps, and agree on $V_j \cap V_{j'}$. We can thus glue them all together to obtain the desired morphism $Y \to \widetilde{X}$. □

4. Projective curves

The main utility of normalization for us is that it provides a method of desingularizing curves, and we see that it preserves projectivity.

Theorem 4.1. *The normalization of a projective curve is a nonsingular projective curve.*

Proof. Let C be the projective curve, and $\widetilde{C}$ its normalization. Nonsingularity of $\widetilde{C}$ is immediate from Corollary 3.3, while we know that $\widetilde{C}$ is quasiprojective from Theorem 2.1. We claim that if we have $\widetilde{C} \subseteq \mathbb{P}^n$, it must be closed, so that $\widetilde{C}$ is projective, as desired. Let Y be the closure of $\widetilde{C}$ in $\mathbb{P}^n$. Given $P \in Y$, let U be an affine neighborhood of P in Y, and $\widetilde{U}$ its normalization. Then since $\widetilde{U}$ is a nonsingular curve and C is projective, by Theorem 1.5 the birational map $\widetilde{U} \dashrightarrow C$ induced by

$$\widetilde{U} \to U \hookrightarrow Y \dashrightarrow \widetilde{C} \to C$$

extends to a morphism $\widetilde{U} \to C$.

Because $\widetilde{U}$ is normal, it follows from Proposition 3.11 that this morphism factors through $\widetilde{C} \to C$. By construction, the induced morphism $\widetilde{U} \to \widetilde{C} \hookrightarrow Y$ agrees with the composed morphism $\widetilde{U} \to U \hookrightarrow Y$ on an open subset, and hence on all of $\widetilde{U}$. But by surjectivity of normalization, there is some $\widetilde{P} \in \widetilde{U}$ mapping to $P \in U$, and if we let Q be its image in $\widetilde{C}$, we conclude $Q = P$ in Y, so $P \in \widetilde{C}$. Since P was arbitrary in Y, we conclude that $\widetilde{C} = Y$, and $\widetilde{C}$ is projective, as desired. □

Corollary 4.2. *Let C be a nonsingular curve. Then there is a nonsingular projective curve $\bar{C}$ (necessarily unique) such that C is isomorphic to an open subvariety of $\bar{C}$.*

Proof. By Theorem 2.1, we can realize C as a quasiprojective curve. Let Y be its closure in projective space. By Theorem 4.1, if $\bar{C}$ is the normalization of Y, it is a projective nonsingular curve. Finally, since C is a nonsingular open subset of Y (and using the assertion on restriction to open subsets in Theorem 3.8), the normalization map $\bar{C} \to Y$ is an isomorphism on C, so we have that C is isomorphic to an open subvariety of $\bar{C}$, as desired. □

Corollary 4.3. *Every curve is birational to a unique nonsingular projective curve.*

Proof. The uniqueness is Corollary 1.7. Given any curve C, we know it has a non-empty open subset U of nonsingular points, so applying Theorem 4.2 to U, we can imbed it into a nonsingular projective curve, which is then birational to C. □

We can rephrase what we've done in more abstract language as follows:

Corollary 4.4. *The following categories are equivalent:*

(a) *Projective nonsingular curves, and nonconstant morphisms between them;*
(b) *Curves, and dominant rational maps between them;*
(c) *Finitely generated field extensions of k of transcendence degree 1, and field inclusions between them.*

This is a powerful tool for studying projective nonsingular curves, as we'll see soon.

Another consequence of the normalization construction is the following:

Corollary 4.5. *If C is a projective curve, D is any curve, and $\varphi : C \to D$ a non-constant morphism, then φ is surjective.*

Note that this is a consequence of our claim that projective varieties are complete, but we will give a more basic proof.

Proof. Since φ is non-constant, it must be dominant: indeed, if $\varphi(C)$ is contained in a proper closed subset of D, it must be a single point, since C (and hence $\varphi(C)$) is connected. It thus induces an injection $K(D) \hookrightarrow K(C)$, and since both function fields have transcendence degree 1, we conclude $K(C)$ is algebraic over $K(D)$. Since they are both finitely generated over k, $K(C)$ is finitely generated over $K(D)$, so is a finite extension. We can thus let $\widetilde{D}_C$ be the normalization of D in $K(C)$. Now, $\widetilde{D}_C$ is birational to C, so we have a rational map $\psi : \widetilde{D}_C \dashrightarrow C$ such that $\varphi \circ \psi$ is equal to the normalization map (considered as a rational map). But because $\widetilde{D}_C$ is nonsingular and C is projective, ψ extends to a morphism which must satisfy the same relation that $\varphi \circ \psi$ is equal to the normalization. But the normalization map is surjective, so we conclude that φ is likewise surjective. □

Corollary 4.6. *If $\varphi : C \to D$ is a nonconstant morphism of curves, then $\varphi(C)$ is an open subset of D.*

Proof. Since D has the cofinite topology, it is enough to show that $\varphi(C)$ contains an open subset of D. We may therefore restrict to the sets of nonsingular points of both C and D, so we reduce to the case that C and D are nonsingular. Now let $\bar{C}$ and $\bar{D}$ be the nonsingular projective compactification of C and D provided by Corollary 4.2. Then φ extends to a morphism $\bar{C} \to \bar{D}$, which we know is surjective by Corollary 4.5. But $\bar{C} \smallsetminus C$ consists of a finite set of points, so $\varphi(C)$ contains all but a finite set of points of $\bar{D}$, and in particular of D, so we conclude it is an open subset of D, as desired. □

Remark 4.7. We've already remarked that Corollary 4.5 generalizes to higher-dimensional varieties as the statement that the image of a morphism from a projective variety to a variety is always closed. Corollary 4.6 also generalizes, to the statement that the image of any dominant morphism of varieties contains an open subset of the target. The latter (in a slightly strengthened form) is known as Chevalley's theorem.

References

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2. David Mumford, *The red book of varieties and schemes*, second, expanded ed., Springer, 1999.