DIVISORS ON NONSINGULAR CURVES

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We now begin a closer study of the behavior of projective nonsingular curves, and morphisms between them, as well as to projective space. To this end, we introduce and study the concept of divisors.

1. Morphisms of curves

Suppose $\varphi : X \to Y$ is a nonconstant morphism of curves. Then we know that φ is dominant. The induced field extension $K(Y) \hookrightarrow K(X)$ must be algebraic, since $K(X)$ and $K(Y)$ have the same transcendence degree, and indeed it must be finite, since $K(X)$ is finitely generated over k.

Definition 1.1. The **degree** of a morphism $\varphi : X \to Y$ of curves is 0 if φ is constant, and is $[K(X) : K(Y)]$ otherwise.

In the case that X, Y are projective, nonsingular curves and φ is nonconstant, we already know that φ is necessarily surjective, but we will prove (more accurately, sketch a proof of) a much stronger result.

Definition 1.2. Suppose $\varphi : X \to Y$ is a nonconstant morphism of nonsingular curves, and $P \in X$ any point. The **ramification index** e_P of φ at P is defined as follows: let $t \in \mathcal{O}_{f(P),Y}$ be a local coordinate at P (that is, at element with $\mathrm{ord}_{f(P)}(t) = 1$); then $e_P = \mathrm{ord}_P(\varphi_P^*(t))$.

Observe that e_P is well defined, since φ_P^* is injective, and $e_P \geqslant 1$ always, since $\varphi_P^* t$ must vanish at P.

Definition 1.3. Suppose $\varphi : X \to Y$ is a morphism of nonsingular curves, and $P \in X$ any point. Then P is a **ramification point** of φ if $e_P \geqslant 2$. In this case, we say $\varphi(P)$ is a **branch point** of φ. If $e_P = 1$, we say that P is **unramified**.

Remark 1.4. Conceptually (and in fact precisely, when one is working over $\mathbb{C}$), a ramification point is a critical point of φ (i.e., a point where the derivative of φ vanishes), and a branch point is a critical value. We will discuss a closely related version of this statement after we have introduced differential forms.

The fundamental result in the case that X, Y are projective is:

Theorem 1.5. *Let $\varphi : X \to Y$ be a nonconstant morphism of projective, nonsingular curves, having degree d, and let $Q \in Y$ any point. Then*

$$\sum_{P \in \varphi^{-1}(Q)} e_P = d.$$

Sketch of proof. We use the projectivity hypothesis only to argue that if $V \subseteq Y$ is an affine open neighborhood of Q, then $U := \varphi^{-1}(V)$ is also affine, and furthermore $A(U)$ is a finitely generated $A(V)$-module. Let $\tilde{Y}_X$ be the normalization of Y in $K(X)$; since Y is projective, we know that $\tilde{Y}_X$ is likewise projective. Also, $\tilde{Y}_X$ is necessarily nonsingular, and because $\tilde{Y}_X$ is birational to X, we conclude that $\tilde{Y}_X$ is isomorphic to X, and the isomorphism commutes with φ and the normalization morphism. But we know that the normalization morphism has the property that the preimage of V

is affine, and in fact its coordinate ring is the integral closure of $A(V)$ in $K(X)$. We conclude that U is affine, and $A(U)$ is integral over $A(V)$. But because $A(U)$ is finitely generated as an algebra, it follows from integrality that $A(U)$ is a finitely-generated $A(V)$-module, as desired.

The desired result now follows from a standard result in algebra on the behavior of extensions of Dedekind domains. In our case, the main idea is to show that if we use the notation

$$\mathcal{O}_{Q,X} := \bigcap_{P \in \varphi^{-1}(Q)} \mathcal{O}_{P,X},$$

then $\mathcal{O}_{Q,X}$ is a free module over $\mathcal{O}_{Q,Y}$ of rank d, and then (if $t \in \mathcal{O}_{Q,Y}$ is a local coordinate) to relate $\mathcal{O}_{Q,X}/(\varphi^*t)$ to the various $\mathcal{O}_{P,X}/(\varphi^*t)$ in terms of the e_P. $\qquad\square$

The geometric intuition behind the theorem is that at most points, the morphism φ is a $d:1$ cover, but that at certain points (the ramification points), some of the d sheets come together.

Example 1.6. Suppose $X = Y = \mathbb{P}^1$, and φ is given by $(X_0, X_1) \mapsto (X_0^d, X_1^d)$. Away from $X_0 = 0$, we can normalize so that $X_0 = 1$, and the morphism is the morphism $\mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^d$. We first assume that we are not in the situation that $\operatorname{char} k = p$ and $p | d$. If $c \neq 0$, the preimage of $x = c$ consists of d distinct points, so we see that each of these d points must be unramified. However, over $c = 0$, we have only a single preimage, so the ramification index at 0 must be d. The situation is symmetric for $x_1 \neq 0$, so we find that the ramification points are $(1,0)$ and $(0,1)$, each with ramification index d, and all the other points are unramified.

Now, suppose that $\operatorname{char} k = p$ and $p | d$. Write $d = p^r d'$, with p not dividing d'. In this case, we see that all points are ramified; $(1,0)$ and $(0,1)$ still have ramification index d, while the rest all have ramification index p^r.

Remark 1.7. Theorem 1.5 has a parallel result in classical algebraic number theory, describing how prime ideals in a ring of integers factor when one extends to a larger ring of integers. It is one of the appealing aspects of Grothendieck's theory of schemes that it allows one to phrase a single theorem which simultaneously encompasses both results.

Exercise 1.8. Show that a nonconstant morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is ramified at all points of $\mathbb{P}^1$ if and only if it factors through the Frobenius morphism.

Our next goal is to study the behavior of such morphisms in more detail. In order to do so, we will have to introduce the concepts of divisors and differential forms.

2. Divisors on curves

An important topic in classical algebraic geometry is the study of divisors. They play a crucial role in understanding morphisms to projective space, and will also be important for us in our study of morphisms between curves. We will restrict our treatment to the case of nonsingular curves, although most of the basic definitions generalize rather easily to the case of higher-dimensional nonsingular varieties (with points being replaced by closed subvarieties of codimension 1). We will assume throughout this section that X is a nonsingular curve.

Definition 2.1. A **divisor** D on X is a finite formal sum $\sum_i c_i[P_i]$, where $c_i \in \mathbb{Z}$, and each P_i is a point of X. Given also $D' = \sum_i c_i'[P_i]$, we write $D \geqslant D'$ if $c_i \geqslant c_i'$ for each i. We say D is **effective** if $D \geqslant 0$. The **degree** $\deg D$ of D is defined to be $\sum_i c_i$.

We can define pullbacks of divisors under morphisms as follows:

Definition 2.2. If X, Y are nonsingular curves, and $\varphi : X \to Y$ is a nonconstant morphism, and $D = \sum c_i[Q_i]$ a divisor on Y, we define the **pullback** of D under φ, denoted $\varphi^*(D)$, to be the divisor $\sum_i \sum_{P \in \varphi^{-1}(Q_i)} e_P c_i[P]$ on X.

We then have the following corollary, which is an immediate consequence of Theorem 1.5.

Corollary 2.3. *Let $\varphi : X \to Y$ be a morphism of projective nonsingular curves, of degree d. Then for any divisor D on Y, we have*
$$\deg \varphi^*(D) = d \deg D.$$

Divisors are closely related to the study of rational functions.

Definition 2.4. Given $f \in K(X)^*$, the associated divisor $D(f)$ is $\sum_{P \in X} \operatorname{ord}_P(f)$. A divisor D is **principal** if $D = D(f)$ for some $f \in K(X)^*$.

Note that $D(f)$ is indeed a divisor: f is regular away from a finite number of points, and where f is regular, it only vanishes at a finite numbers of points.

Remark 2.5. The terminology of principal divisor is suggestive, and indeed there is a close relationship between principal divisors and principal ideals. However, we will not discuss this until after we have introduced schemes.

If we stick to projective curves, we find that principal divisors are quite well behaved. In particular:

Proposition 2.6. *A principal divisor on a projective nonsingular curve has degree 0.*

Proof. Let $D(f)$ be a principal divisor on the projective nonsingular curve X. If f is constant, then $D(f) = 0$, so the degree is visibly 0. If f is nonconstant, we have seen that it defines a dominant rational map to $\mathbb{A}^1$, which we may extend to a morphism $f : X \to \mathbb{P}^1$. We claim that $D(f) = f^*([0] - [\infty])$. But this is clear: the morphism $f : X \to \mathbb{P}^1$ is induced by the field inclusion $k(t) \to K(X)$ sending t to f, and $\operatorname{ord}_{(0)} t = 1$, so the definition of ramification index gives us precisely that the part of $D(f)$ with positive coefficients are the ramification indices of zeroes of f. But similarly, $\operatorname{ord}_\infty \frac{1}{t} = 1$, and maps to $\frac{1}{f}$, so the negative part of $D(f)$ is given by $\operatorname{ord}_P(\frac{1}{f}) = -\operatorname{ord}_P(f)$ at points P with $f(P) = \infty$.

The desired statement then follows immediately from Corollary 2.3, since $\deg([0] - [\infty]) = 0$. $\square$

Example 2.7. Consider the case that $X = \mathbb{P}^1$, and write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, with coordinate t on $\mathbb{A}^1$. Then a rational function $f \in K(X)^*$ is a quotient of polynomials $g, h \in k[t]$. At any point $\lambda \in \mathbb{A}^1$, the coefficient of $[\lambda]$ in $D(f)$ is simply the difference of the orders of vanishing of g and h at $t = \lambda$. On the other hand, at ∞ we have that $\frac{1}{t}$ is a local coordinate, and one then checks that the coefficient of $[\infty]$ in $D(f)$ is equal to $\deg h - \deg g$. The sum of the coefficients of points on $\mathbb{A}^1$ is $\deg g - \deg h$, so we see that the degree of $D(f)$ is 0, as asserted by Proposition 2.6.

We use divisors to study rational functions on a curve by considering all functions which vanish to certain prescribed orders at some points, and are allowed to have poles of certain orders at others. Formally, we have the following definition.

Definition 2.8. Given a divisor D on X, define
$$\mathcal{L}(D) = \{f \in K(X)^* : D(f) + D \geqslant 0\} \cup \{0\}.$$

Thus, if $D = \sum_i c_i [P_i] - \sum_j d_j [Q_j]$ where $c_i, d_j > 0$, and $P_i \neq Q_j$ for any i, j, then $\mathcal{L}(D)$ is the space of all rational functions which vanish to order at least d_j at each Q_j, but are allowed to have poles of order at most c_i at each P_i. This is visibly a k-vector space, and we will next prove that it is finite-dimensional when X is projective.

Lemma 2.9. *Given a divisor D on X, and $P \in X$ any point, the quotient space $\mathcal{L}(D)/\mathcal{L}(D - [P])$ has dimension at most 1.*

Proof. First observe that $\mathcal{L}(D - [P])$ is indeed a subspace of $\mathcal{L}(D)$, consisting precisely of those rational functions vanishing to order (possibly negative) strictly greater than required to be in $\mathcal{L}(D)$. The quotient vector space thus makes sense. If $\mathcal{L}(D - [P]) = \mathcal{L}(D)$ (which can certainly occur), there is nothing to prove. On the other hand, if $\mathcal{L}(D - [P]) \subsetneq \mathcal{L}(D)$, let f be in $\mathcal{L}(D) \smallsetminus \mathcal{L}(D - [P])$, and let t be a local coordinate on X at P. Then we know that $f = t^e g$ for some $e \in \mathbb{Z}$, and some g regular and nonvanishing in a neighborhood of P. Since $f \notin \mathcal{L}(D - [P])$, we see that $-e$ must be the coefficient of $[P]$ in D. We claim that f spans $\mathcal{L}(D)/\mathcal{L}(D - [P])$. Indeed, for any other $f' \in \mathcal{L}(D)$, we can write $f' = t^{e'} g'$ with g' regular and nonvanishing at P, and we must have $e' - e \geqslant 0$, so $e' \geqslant e$. Then we observe that

$$f' - \frac{t^{e'-e}(P)g'(P)}{g(P)}f$$

vanishes to order strictly greater than e at P, so is therefore in $\mathcal{L}(D - [P])$, proving our claim and the lemma. $\qquad\square$

Corollary 2.10. *For any divisor D on a projective nonsingular curve X, we have that $\mathcal{L}(D)$ is finite-dimensional over k, and in fact*

$$\dim_k \mathcal{L}(D) \leqslant \deg D + 1.$$

Proof. We see immediately from Proposition 2.6 that if $\deg D < 0$, then $\mathcal{L}(D) = 0$. The statement then follows immediately from Lemma 2.9 by induction on $\deg D$. $\qquad\square$

This dimension will be very important to us, so we give it its own notation.

Notation 2.11. We denote by $\ell(D)$ the dimension of $\mathcal{L}(D)$ over k.

Example 2.12. Again consider the case $X = \mathbb{P}^1$, continuing with the notation of Example 2.7, and let $D = d[\infty]$. Then if $f \in K(X)^*$ is written as $\frac{g}{h}$, where g, h have no common factors, we see that for f to be in $\mathcal{L}(D)$, we must have h constant, since we cannot have any poles away from ∞. Then we have $\deg h - \deg g = -\deg g \leqslant -d$, so we conclude that f must be a polynomial of degree at most d. Conversely, any polynomial of degree less than or equal to d is in $\mathcal{L}(D)$, so we see that $\ell(D) = d + 1$. In particular, sometimes the bound of Corollary 2.10 is achieved.

3. Linear equivalence and morphisms to projective space

Closely related to the study of divisors and rational functions is the study of morphisms to projective space. In this section, we suppose throughout that X is a nonsingular projective curve.

Suppose we have a morphism $\varphi : X \to \mathbb{P}^n$, which we assume to be **non-degenerate**, meaning that $\varphi(X)$ is not contained in any hyperplane H of $\mathbb{P}^n$. We then see that for any such H, we have $H \cap \varphi(X)$ a proper closed subset of $\varphi(X)$, hence a finite set of points. In fact, there is a natural way to associate an effective divisor on X to $H \cap \varphi(X)$. Suppose $P \in X$ such that $\varphi(P) \in H$, and choose i such that $\varphi(P) \in U_i = \mathbb{P}^n \smallsetminus Z(X_i)$. If $H = Z(\sum_j c_j X_j)$, then we identify U_i with $\mathbb{A}^n$ in the usual way by setting coordinates $y_j = \frac{X_j}{X_i}$ for $j \neq i$, and on U_i we have $Z(H) = Z(\sum_j c_j y_j)$, where $y_i = 1$. Then $\sum_j c_j y_j$ is a regular function on U_i, so $\varphi^{-1}(\sum_j c_j y_j)$ is regular at P, and we take its order at P to determine the coefficient of P in the divisor associated to $H \cap \varphi(X)$. One checks easily that this is independent of the choice of i and of the equation for H (which is unique up to scaling).

Notation 3.1. If $\varphi : X \to \mathbb{P}^n$ is a nondegenerate morphism, and $H \subseteq \mathbb{P}^n$ a hyperplane, we denote by $\varphi^*(H)$ the effective divisor on X associated to $H \cap \varphi(X)$.

Obviously, $\varphi^*(H)$ depends on H. However, if we choose a different hyperplane H', we find that $\varphi^*(H)$ and $\varphi^*(H')$ are closely related.

Proposition 3.2. *Given* $\varphi : X \to \mathbb{P}^n$ *a nondegenerate morphism, and* $H, H' \subseteq \mathbb{P}^n$ *two hyperplanes, then* $\varphi^*(H) - \varphi^*(H') = D(f)$ *for some rational function* f *on* X.

Proof. The basic idea is that if $H = Z(\sum_i c_i X_i)$, and $H' = Z(\sum_i c'_i X_i)$, then even though the defining equations do not give functions on $\mathbb{P}^n$, the quotient $\frac{\sum_i c_i X_i}{\sum_i c'_i X_i}$ defines a rational function on $\mathbb{P}^n$, which by non-degeneracy is regular and non-vanishing on a nonempty open subset of $\varphi(X)$, and thus pulls back under φ to give a rational function f on X. We need only verify that $D(f) = \varphi^*(H) - \varphi^*(H')$.

However, following the notation of the above discussion, if the ith coordinate of P is non-zero, we can divide through both the numerator and denominator by X_i and find

$$\frac{\sum_j c_j X_j}{\sum_j c'_j X_j} = \frac{\sum_j c_j y_j}{\sum_j c'_j y_j},$$

so the definition of $D(f)$ is visibly equal to $\varphi^*(H) - \varphi^*(H')$. $\qquad\square$

This motivates the following definition:

Definition 3.3. Two divisors D, D' on X are **linearly equivalent** if $D - D' = D(f)$ for some rational function f on X.

From Proposition 2.6, we see immediately:

Corollary 3.4. *Two linearly equivalent divisors have the same degree.*

We can thus define:

Definition 3.5. The **degree** of a nondegenerate morphism $\varphi : X \to \mathbb{P}^n$ is $\deg \varphi^*(H)$ for any hyperplane $H \subseteq \mathbb{P}^n$.

Remark 3.6. It is a fact, although we will not prove it this quarter, that if $\varphi(X)$ is injective, then the degree is equal to the number of points in $\varphi(X) \cap H$ for a "sufficiently general" hyperplane H. That is, if H is not too special, all the points of $\varphi^*(H)$ will have multiplicity 1.

There is one situation of overlap between this definition and our earlier definition of degree for morphisms between curves. We verify that the two definitions agree in this case.

Proposition 3.7. *Let* $\varphi : X \to \mathbb{P}^1$ *be a nonconstant morphism. Then* $\deg \varphi^* H = [K(X) : K(\mathbb{P}^1)]$, *where* H *is any hyperplane (that is to say, point) in* $\mathbb{P}^1$.

Proof. This is not obvious from the definitions, but it follows easily from Theorem 1.5. Indeed, we see immediately from the definitions that

$$\varphi^* H = \sum_{P \in \varphi^{-1}(H)} e_P[P],$$

so the desired identity follows. $\qquad\square$

Linear equivalence also arises in the spaces $\mathcal{L}(D)$. If $f \in K(X)^*$ is in $\mathcal{L}(D)$, then instead of looking at the zeroes and poles of f as a rational function, we could ask what its "extra vanishing" is as an element of $\mathcal{L}(D)$; that is, we could look at the divisor $D(f) + D$, which is effective by definition. We have:

Proposition 3.8. *Given a divisor* D, *the set* $D(f) + D$ *for nonzero* $f \in \mathcal{L}(D)$ *is precisely the set of effective divisors linearly equivalent to* D.

Proof. Suppose D' is effective, and linearly equivalent to D. Then by definition, there is some $f \in K(X)^*$ with $D' - D = D(f)$, or equivalently, $D' = D(f) + D$. Thus $f \in \mathcal{L}(D)$, and we get one inclusion. Conversely, we have already observed that $D(f) + D$ is effective by definition if $f \in \mathcal{L}(D)$ is nonzero, but it is also visibly linearly equivalent to D. $\qquad\square$

We can thus think of the following definition in terms of families of effective, linearly equivalent divisors:

Definition 3.9. A **linear series** is a vector subspace of $\mathcal{L}(D)$ for some D. A linear series is **complete** if it is equal to all of $\mathcal{L}(D)$. We say that the linear series has **degree** d and **dimension** n if $\deg D = d$, and the subspace has dimension $n + 1$.

Associated to a linear series we get a family of effective divisors, but the linear series language is useful because the condition of being a subspace is easier to understand than the corresponding condition on the associated divisors. The discrepancy of 1 in the dimension terminology will be explained shortly.

We now return to considering morphisms to projective space. As above, we suppose we have $\varphi : X \to \mathbb{P}^n$. We observe that the associated family of divisors $\varphi^*(H)$ for hyperplanes H in $\mathbb{P}^n$ have the property that there is no point $P \in X$ such that $[P]$ appears with positive coefficients in every $\varphi^*(H)$: indeed, we can choose H to be any hyperplane in $\mathbb{P}^n$ such that $\varphi(P) \notin H$, and then the coefficient of $[P]$ in $\varphi^*(H)$ will be 0.

Definition 3.10. Given a linear series $V \subseteq \mathcal{L}(D)$, a point $P \in X$ is a **basepoint** of V if for all nonzero $f \in V$, the coefficient of $[P]$ in $D(f) + D$ is strictly positive. The linear series V is **basepoint-free** if no $P \in X$ is a base point of V.

Lemma 3.11. *Suppose that D and D' are linearly equivalent divisors on a projective nonsingular curve X. Then there is an isomorphism $\alpha : \mathcal{L}(D) \to \mathcal{L}(D')$, unique up to scaling, such that $D(\alpha(f)) + D' = D(f) + D$ for all nonzero $f \in \mathcal{L}(D)$.*

Proof. By definition, there is some $g \in K(X)^*$ with $D - D' = D(g)$. Because X is projective, only constant functions have $D(g) = 0$, we see that g is unique up to scaling by a (nonzero) constant. Multiplication by g then defines the desired isomorphism $\mathcal{L}(D) \to \mathcal{L}(D')$. $\qquad\square$

Definition 3.12. We say that two linear series $V \subseteq \mathcal{L}(D)$ and $V' \subseteq \mathcal{L}(D')$ are **equivalent** if D is linearly equivalent to D' and V is mapped to V' under the isomorphism of Lemma 3.11.

Proposition 3.13. *Linear series V and V' are equivalent if and only if the sets of effective divisors $\{D(f) + D : f \in V \smallsetminus \{0\}\}$ and $\{D(f) + D' : f \in V' \smallsetminus \{0\}\}$ are equal.*

Proof. It is clear from the definition and Lemma 3.11 that if V and V' are equivalent, the sets of effective divisors are equal. Conversely, if the sets of effective divisors have any elements in common, we immediately conclude that D is linearly equivalent to D', and because the isomorphism α of Lemma 3.11 doesn't change the corresponding effective divisors $D(f) + D$, and $D(f) + D$ determines f up to nonzero scalar, we conclude that if the two sets of divisors are equal, then α must map V into V'. $\qquad\square$

The main theorem relating linear series to morphisms to projective space is the following:

Theorem 3.14. *Let $V \subseteq \mathcal{L}(D)$ be a basepoint-free linear series on X of dimension n and degree d. Then V is associated to a nondegenerate morphism $\varphi : X \to \mathbb{P}^n$, unique up to linear change of coordinate on $\mathbb{P}^n$, such that the set $D(f) + D$ for nonzero $f \in V$ is equal to the set of $\varphi^*(H)$ for hyperplanes $H \subseteq \mathbb{P}^n$. In particular, φ has degree d.*

Moreover, this construction gives a bijection between equivalence classes of linear series of dimension n and degree d on X, and nondegenerate morphisms $\varphi : X \to \mathbb{P}^n$ of degree d, up to linear change of coordinates.

Lemma 3.15. *Suppose a morphism $\varphi : X \to \mathbb{P}^n$ is described on an open subset $U \subseteq X$ by a tuple $(f_0, \ldots, f_n)$ of functions regular and not simultaneously vanishing on U. Then φ is nondegenerate if and only if the f_i are linearly independent in $K(X)$ over k, and if $H \subseteq \mathbb{P}^n$ is the hyperplane $Z(\sum_i c_i X_i)$ for some $c_i \in k$, and*

$$D = - \sum_{P \in X} \min_i(\operatorname{ord}_P(f_i))[P],$$

then

$$\varphi^*(H) = D(\sum_i c_i f_i) + D.$$

Proof. The assertion of nondegeneracy is clear, since if $\varphi(U)$ were contained in a hyperplane, we would obtain a linear dependence among the f_i, and conversely. To check the desired identity of divisors, fix $P \in X$, and suppose the ith coordinate of $\varphi(P)$ is nonzero; then the coefficient of $[P]$ in $\varphi^*(H)$ is defined to be the order of vanishing at P of $\varphi^*(\sum_j c_j \frac{X_j}{X_i})$. Let e be the coefficient of $[P]$ in D. Then each f_j has order at least $-e$ at P by hypothesis, and in fact at least one f_j has order exactly $-e$.

We know that φ is defined at P by multiplying all the f_j by t^e, for a local coordinate t at P, so that near P, the morphism φ is given by $(t^e f_0, \ldots, t^e f_n)$, and so we see in particular that f_i has minimum order $-e$ at P. Furthermore, near P we have

$$\varphi^*(\frac{X_j}{X_i}) = \frac{t^e f_j}{t^e f_i} = \frac{f_j}{f_i},$$

so the coefficient of $[P]$ in $\varphi^*(H)$ is

$$\operatorname{ord}_P(\sum_j c_j \frac{f_j}{f_i}) = \operatorname{ord}_P(\sum_j c_j f_j) + e,$$

proving the desired statement. $\square$

Proof of Theorem 3.14. If we choose a basis $f_0, \ldots, f_n$ for V, by linear independence and Lemma 3.15, the tuple $(f_0, \ldots, f_n)$ defines a nondegenerate morphism $\varphi : X \to \mathbb{P}^n$, which satisfies the desired relationship. Now, in constructing φ, we chose the basis $\{f_i\}$ for V. It is clear that change of basis corresponds precisely to to modifying φ by composing $\mathbb{P}^n$ by the corresponding linear change of coordinates. We thus wish to prove that aside from this ambiguity, φ is uniquely determined by the set of $\varphi^*(H)$. But we see that if $\varphi : X \to \mathbb{P}^n$ is any nondegenerate morphism, we obtain effective divisors $D_0, \ldots, D_n$ with $D_i := \varphi^*(Z(X_i))$, which by hypothesis are each of the form $D(g_i) + D$ for some $g_i \in \mathcal{L}(D)$. But if φ is given by $(f_0, \ldots, f_n)$, and we let $D' = - \sum_{P \in X} \min_i(\operatorname{ord}_P(f_i))$, then by Lemma 3.15 we see that $D(f_i) + D' = D(g_i) + D$ for each i, so $D(f_i/g_i) = D - D'$. If we replace each g_i by $\frac{f_0}{g_0} g_i$, then we don't change φ, and we have $D(f_i) = D(g_i)$, so the f_i and g_i are each related by nonzero scalars, and we conclude the desired uniqueness.

Now, because the associated morphisms φ are characterized by the effective divisors $D(f) + D$, we immediately see that equivalent linear series yield the same (equivalence classes of) morphisms to projective space. For the final bijectivity assertion, it is thus enough to show that every nondegenerate morphism to projective space arises in the manner described. However, every morphism $\varphi : X \to \mathbb{P}^n$ is described on some open subset U by a tuple $(f_0, \ldots, f_n)$ of regular functions on U, and from Lemma 3.15 we then see that if we set D as in the lemma statement, the f_i span a linear series in $\mathcal{L}(D)$ which satisfies the desired conditions. $\square$