THE RIEMANN-ROCH AND RIEMANN-HURWITZ THEOREMS

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We state without proof the Riemann-Roch theorem, and give some basic applications, including a proof of the Riemann-Hurwitz theorem. We use throughout the convention that all curves are projective and nonsingular.

1. The Riemann-Roch theorem

With all the preliminaries out of the way, we can state one of the most fundamental theorems in the study of algebraic curves, the Riemann-Roch theorem.

Theorem 1.1. *Let X be a curve of genus g, and D a divisor on X of degree d. Then*

$$\ell(D) - \dim_k \Omega(-D) = d + 1 - g.$$

One immediate consequence is:

Corollary 1.2. *Let ω be a non-zero rational differential form on a curve X of genus g. Then $D(\omega)$ has degree $2g-2$.*

Proof. We apply the Riemann-Roch theorem in the case that $D = D(\omega)$. We know that $\Omega(-D)$ is isomorphic to $\mathcal{L}(D(\omega) - D)$, so $\Omega(D(\omega))$ is isomorphic to $\mathcal{L}(0) = k$. On the other hand, $\mathcal{L}(D(\omega))$ is isomorphic to $\Omega(0)$, so $\ell(D(\omega)) = g$. We conclude that

$$g - 1 = \deg D(\omega) + 1 - g,$$

giving the desired identity. □

However, the Riemann-Roch theorem has a wide range of applications. For instance, we see that every curve of genus 0 is isomorphic to $\mathbb{P}^1$.

Exercise 1.3. Let X be a curve. Suppose X has a divisor D of degree $d > 0$, such that $\ell(D) = d+1$. Prove that $X \cong \mathbb{P}^1$. Conclude that if X has genus 0, then $X \cong \mathbb{P}^1$. Hint: for the first part, begin with the case that $d = 1$.

The Riemann-Roch theorem also tells us that for divisors D of high enough degree, there is no difficulty in understanding $\ell(D)$:

Corollary 1.4. *Let X be a curve, and D a divisor of degree $d > 2g-2$. Then $\ell(D) = d + 1 - g$.*

Proof. If ω is a rational differential form, we know that $\Omega(-D) \cong \mathcal{L}(D(\omega) - D) = 0$, since $\deg D(\omega) - D < 0$. The statement then follows from the Riemann-Roch theorem. □

We also sketch another direction of application, which we don't have time to pursue in detail.

Remark 1.5. For a divisor D, and an effective divisor D', we know that the codimension of $\mathcal{L}(D - D')$ inside of $\mathcal{L}(D)$ is at most $\deg D'$. In particular, if $D' = [P]$ for some $P \in X$, then $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - [P])$ is 0 or 1. We see that the condition that the complete linear series corresponding to $\mathcal{L}(D)$ be base-point-free is exactly the same as requiring that $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - [P]) = 1$ for all P. Now, suppose $\mathcal{L}(D)$ is base-point-free, so that it defines a morphism $\varphi : X \to \mathbb{P}^r_k$, where $\ell(D) = r + 1$.

If $\varphi(P) = \varphi(Q)$ for some $P \neq Q$, then any hyperplane $H \subseteq \mathbb{P}^r_k$ which contains P also contains Q, so we see from our construction of φ that $\mathcal{L}(D - [P]) = \mathcal{L}(D - [Q])$, and hence both are equal to $\mathcal{L}(D - [P] - [Q])$, so $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - [P] - [Q]) = 1$. Conversely, if $\varphi(P) \neq \varphi(Q)$, then there exist hyperplanes containing $\varphi(P)$ but not $\varphi(Q)$, and we conclude that $\mathcal{L}(D - [P] - [Q]) \subsetneq \mathcal{L}(D - [P])$, so $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - [P] - [Q]) = 2$. Thus, we see that φ is injective if and only if $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - [P] - [Q]) = 2$ for all $P \neq Q$ on X.

Now, the idea is that if φ is injective, it should define an isomorphism onto its image exactly when the induced maps on tangent spaces are also injective, which should in turn correspond to the condition that $\mathcal{L}(D - 2[P]) \subsetneq \mathcal{L}(D - [P])$ for all $P \in X$. If we believe this, we conclude the following criterion: $\mathcal{L}(D)$ induces a morphism $\varphi : X \to \mathbb{P}^r_k$ which is an isomorphism onto its image if and only for all $P, Q \in X$ (not necessarily distinct), we have $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - [P] - [Q]) = 2$.

In conjunction with the Riemann-Roch theorem, this is a useful tool. For instance, it follows immediately from Corollary 1.4 that if $\deg D > 2g$, then $\mathcal{L}(D)$ induces a morphism which is an isomorphism onto its image.

In particular, we see that if X has genus 1, and D is any divisor of degree 3, then $\mathcal{L}(D)$ gives a morphism $X \to \mathbb{P}^2_k$ of degree 3 which is an isomorphism onto its image. This is the well-known fact that every curve of genus 1 can be realized as a curve in the plane of degree 3.

Remark 1.6. The proof of the Riemann-Roch theorem is rather difficult, but we can at least explain why the statement is reasonable. We first observe that because $\ell(0) = 1$ and $\dim_k \Omega(0) = g$ by definition, the desired statement holds for $D = 0$. Now, given any divisor D and $P \in X$, we know that $\ell(D + P) - \ell(D)$ and $\dim_k \Omega(-D) - \dim_k \Omega(-D - P)$ are both equal to 0 or 1, and what we want to show is that one is equal to 1 if and only if the other is equal to 0 – the theorem then follows by induction on the number of points in D (counted in terms of the absolute value of their coefficients).

To see why this makes sense, let V_P be the one-dimensional vector space described as follows: if c is the coefficient of $[P]$ in D, set

$$V_P = (\{0\} \cup \{f \in K(X)^* : \operatorname{ord}_P(f) \geqslant -c - 1\}) \,/\, (\{0\} \cup \{f \in K(X)^* : \operatorname{ord}_P(f) \geqslant -c\}) \,.$$

Then clearly we have an exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + P) \to V_P,$$

and this is surjective on the right exactly when $\ell(D+P) - \ell(D) = 1$. On the other hand, if we take the dual of the natural injection $\Omega(-D-P) \hookrightarrow \Omega(-D)$, we obtain a surjection $\Omega(-D)^* \twoheadrightarrow \Omega(-D-P)^*$.

We will define a map $V_P \to \Omega(-D)^*$ using the idea of residues, for which we only need the simplest case: if ω is a rational differential form with at worst a simple pole at P, then we can write $\omega = g dt$, where t is a local coordinate at t and $\operatorname{ord}_P(g) \geqslant -1$. We then set $\operatorname{res}_P(\omega) = (tg)(P)$. In this case, it is very straightforward to check the following basic properties:

(i) $\operatorname{res}_P(\omega)$ doesn't depend on the choice of t;
(ii) $\operatorname{res}_P(\omega)$ is k-linear in ω;
(iii) $\operatorname{res}_P(\omega) = 0$ if and only if ω is in fact regular at P.

Now, given $f \in K(X)^*$ with $\operatorname{ord}_P(f) \geqslant -c - 1$, and $\omega \in \Omega(-D - P)$ nonzero, we note that $f\omega$ has at worst a simple pole at P, so $\operatorname{res}_P(f\omega)$ gives an element of k. If in fact $\operatorname{ord}_P(f) \geqslant -c$, we get that $\operatorname{res}_P(f\omega) = 0$, so we see that we have constructed the desired k-linear map $V_P \to \Omega(-D)^*$. We see that the resulting sequence

$$V_P \to \Omega(-D)^* \to \Omega(-D - P)^* \to 0$$

is exact: indeed, it is clear that the image from the left is contained in the kernel, but both of these are either 0 or 1-dimensional, and we see that the map on the left is nonzero if and only if $\Omega(-D)$ contains some ω with order exactly c at P, if and only if the second map has nonzero kernel.

We thus have a sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to V_P \to \Omega(-D)^* \to \Omega(-D-P)^* \to 0,$$

which is exact everywhere except possibly at V_P, and in order to prove the Riemann-Roch theorem, it is enough to prove exactness at V_P. Now, it is not hard to see why we have a complex at V_P: if we have $f \in \mathcal{L}(D+P)$ and $\omega \in \Omega(-D)$, then $f\omega$ is regular except possibly for a simple pole at P, and then the residue theorem (for algebraic curves) implies that $\operatorname{res}_P(f\omega) = 0$. The core of the proof of the Riemann-Roch theorem is then to prove that the image of the map to V_P contains the kernel of the map to $\Omega(-D)^*$.

2. The Riemann-Hurwitz theorem

We can now prove the Riemann-Hurwitz theorem (also sometimes called the Hurwitz theorem), which relates the ramification of a separable morphism to its degree, as well as the genus of the curves.

Theorem 2.1. *Let $\varphi : X \to Y$ be a separable morphism of curves, of degree d. Let g_X and g_Y be the genus of X and Y respectively. Then we have*

$$2g_X - 2 \geqslant d(2g_Y - 2) + \sum_{P \in X} (e_P - 1),$$

with equality if and only if φ is tamely ramified.

Proof. Let ω be a nonzero rational differential form on Y, such that $\varphi^*\omega$ is also nonzero; this exists by separability. By Corollary 1.2, we know that $D(\omega)$ has degree $2g_Y - 2$, and $D(\varphi^*\omega)$ has degree $2g_X - 2$. We claim that

$$D(\varphi^*\omega) \geqslant \varphi^* D(\omega) + \sum_{P \in X} (e_P - 1)[P],$$

with equality if and only if φ is tamely ramified. The desired result then follows by taking degrees, using that $\deg \varphi^* D(\omega) = d \deg D(\omega)$. Given $P \in X$, let s be a local coordinate at P, and t be a local coordinate at $\varphi(P)$, and write $\omega = ft^n dt$ for some $f \in \mathcal{O}^*_{\varphi(P),Y}$, and n is by definition the coefficient of $[P]$ in $D(\omega)$. Then the coefficient of $[P]$ in $\varphi^*\omega = (\varphi^* ft^n)\varphi^* dt$ is its order of vanishing at P, which is the sum of $\nu_P \varphi^* ft^n$ and the order of vanishing of $\varphi^* dt$ at P. But $\nu_P \varphi^* ft^n = n\nu_P \varphi^* t = e_P n$ by definition of the ramification index, which is also equal to the coefficient of $[P]$ in $\varphi^* D(ft^n)$, which in turn is by definition the coefficient of $[P]$ in $\varphi^* D(\omega)$. On the other hand, the order of vanishing of $\varphi^* dt$ at P is at least $e_P - 1$, with equality if and only if φ is tamely ramified at P. This proves the claim, and the theorem. □

One immediate consequence is the following:

Corollary 2.2. *Let $\varphi : X \to Y$ be a tamely ramified morphism of curves. Then $\sum_{P \in X}(e_P - 1)$ is even.*

Example 2.3. Consider the morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given on $\mathbb{A}^1$ by $x \mapsto x^{2p} - x$. Then one checks that this is unramified on $\mathbb{A}^1$, and ramified to order $2p$ at ∞; thus, Corollary 2.2 is false without the tame ramification hypothesis.

Corollary 2.4. *Let $\varphi : X \to Y$ be a separable morphism of curves of genus g_X and g_Y, respectively. Then $g_X \geqslant g_Y$, and equality is only possible if $g_Y = 0, 1$ or $d = 1$.*

Proof. For $g_Y = 0$, there is nothing to prove. For $g_Y = 1$, the righthand side of the Riemann-Hurwitz formula is nonnegative, so $g_X \geqslant 1$. But for $g_Y > 1$, we have $g_X - 1 \geqslant d(g_Y - 1) \geqslant g_Y - 1$, with equality only possible if $d = 1$, so we conclude the desired statement. □

Remark 2.5. In fact, Corollary 2.4 holds for arbitrary nonconstant morphisms: one reduces to the separable case using that any such morphism factors as a composition of Frobenius morphisms followed by a separable morphism, and noting that the Frobenius morphisms leave the genus unchanged.

Remark 2.6. Given our definition of genus, it is natural to wonder how easy it is to see that it agrees with the topological notion of genus in the case $k = \mathbb{C}$. One way to verify that these indeed agree is to prove the Riemann-Hurwitz theorem also using the topological definition for branched covers of surfaces, and then use that every curve has a nonconstant morphism to $\mathbb{P}^1_k$ to conclude that the definitions of genus must coincide.