ABSTRACT VARIETIES VIA ATLASES

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In this expository note we describe how abstract algebraic varieties over an algebraically closed field may be defined rigorously via an atlas definition analogous to the usual definition of differential manifolds. The restriction to algebraically closed fields allows us to use the naive notions of the underlying spaces and their morphisms; we thus fix throughout an algebraically closed field k.

1. Prevarieties

In the definition of manifolds, one imposes a condition that every point have a neighborhood which is isomorphic (in some appropriate sense) to an open subset of $\mathbb{R}^n$. This is not possible for varieties, for two reasons. The first is that we do not want to restrict ourselves to smooth varieties, so even for complex varieties we will not necessarily obtain topological manifolds. However, even if we wished to restrict to smooth varieties, we could still not ask for our varieties to be locally isomorphic to an open subset of affine space, because algebraic maps are fundamentally much more rigid than differentiable maps, and it is simply not the case that a smooth variety has an open cover by varieties which can be thought of as open subvarieties of affine space. We therefore allow our varieties instead to be covered by open subsets which are isomorphic to affine varieties.

Asking for a variety to have an open cover by affine varieties may at first be counterintuitive – in analogy with manifold theory, why not allow the open subsets to be isomorphic to open subsets of affine varieties? However, we have just seen that every open subset of an affine variety has an open cover by affine varieties, so in fact this would not be any different, and we conserve words by imposing that our open cover consist of affine varieties.

To summarize, we will construct general abstract varieties by gluing together affine varieties along open subsets, with the restriction that the gluing maps must be algebraic. We have:

Definition 1.1. A **prevariety** X over k is an irreducible topological space, together with an open cover $U_1, \dots, U_m$, and a collection of homeomorphisms $\varphi_i : X_i \overset{\sim}{\to} U_i$, where each $X_i \subseteq \mathbb{A}^{n_i}$ is an affine variety equipped with the Zariski topology, and we require that every **transition map**

$$\varphi_{i,j} : \varphi_i^{-1}(U_i \cap U_j) \overset{\varphi_j^{-1}\circ\varphi_i}{\overset{\sim}{\to}} \varphi_j^{-1}(U_i \cap U_j)$$

is a morphism. We say that each map $\varphi_i : X_i \to U_i \subseteq X$ is a **chart**, and the collection of charts is an **atlas**.

Remarks 1.2. Since $\varphi_{i,j}^{-1} = \varphi_{j,i}$, the transition maps are necessarily isomorphisms. One often thinks of a prevariety as being obtained from the collection of affine varieties X_i by gluing together open subsets along the isomorphisms given by the transition maps.

We have not yet defined varieties because we haven't yet imposed the condition analogous to the Hausdorff condition for a manifold. We will revisit this shortly.

One can vary the definition a bit by defining a notion of equivalence of atlases and speaking of a prevariety as a set with an equivalence class of atlases, or alternatively, by requiring an atlas to be maximal. Either of these options removes the "dependence on choice" of the atlas, but at this point it is not clear whether it would be any less technical to simply do what modern algebraic geometers do, which is to work with sheaves.

Example 1.3. Any affine variety is a prevariety, with an atlas consisting of a single chart.

Lemma 1.4. *Any open subset of a prevariety has a natural structure of a prevariety. Any irreducible closed subset of a prevariety has a natural structure of a prevariety.*

Proof. If X is a prevariety with a chosen atlas, and $U \subseteq X$ is open, then we have proved that we can cover each $\varphi_i^{-1}(U_i \cap U) \subseteq X_i$ with open subsets isomorphic to affine varieties (and indeed finitely many, since X_i is a Noetherian topological space), and passing to this refined cover and using the charts induced by the original φ_i, we get an atlas for U. In the closed case, since any irreducible closed subset of X_i is again an affine variety, we don't even need to refine the covers. □

Combining the example with the first part of the lemma, we conclude:

Corollary 1.5. *Any quasiaffine variety is a prevariety.*

The lemma also motivates the definition:

Definition 1.6. If X is a prevariety, a **subprevariety** of X is an irreducible closed subset of an open subset of X.

By Lemma 1.4, a subprevariety has the structure of a prevariety.

Example 1.7. Suppose we have $U_1 = U_2 = \mathbb{A}^1_k$, and set $U = \mathbb{A}^1_k \smallsetminus (0)$. We consider two different possibilities for gluing U_1 to U_2 along U to obtain a prevariety.

The first is to let X be the union of U_1 and U_2 glued along U, where we identify $U \subseteq U_1$ and $U \subseteq U_2$ simply by the identity map. In this case, X is almost the same as $\mathbb{A}^1_k$, except that now it has two copies of the origin instead of one. In the usual real or complex topology, this satisfies the conditions to be a manifold except that it is not Hausdorff.

On the other hand, we could identify the points of $U \subseteq U_1$ with $U \subseteq U_2$ via the inversion map $t \mapsto 1/t$. As we will see later, this is one way of describing the projective line $\mathbb{P}^1_k$. In the usual real or complex topology, this will give a manifold, and will in fact be compact. We can picture that by adding in U_2 we have compactified U_1 – because our transition map is $t \mapsto 1/t$, the origin of U_2 becomes the "point at infinity" of U_1.

2. Morphisms

We next define morphisms of prevarieties, and in particular we will see when two prevarieties are isomorphic, helping us remove the dependence on choice of atlas. Following the approach of [1] for quasiaffine varieties, we first define regular functions.

Definition 2.1. If X is a prevariety with a given atlas, and $U \subseteq X$ is open, a function $f : U \to k$ is **regular** if for all i, the induced function

$$f \circ \varphi_i : \varphi_i^{-1}(U \cap U_i) \to k$$

is regular (in the sense defined for quasiaffine varieties).

It is not immediately obvious from the definitions, but in fact this concept of regular function is completely compatible with the definition for quasiaffine varieties, and also with our concept of subprevarieties. These compatibilities are laid out in the exercises that follow.

Exercise 2.2. If X is a prevariety with atlas $\{\varphi_i : X_i \to U_i\}$, and $U \subseteq U_i$ for some i, then a function $f : U \to k$ is regular if and only if $f \circ \varphi_i$ is regular in the quasiaffine sense on $\varphi_i^{-1}(U) \subseteq X_i$.

Exercise 2.3. If X is a prevariety, and $V \subseteq U \subseteq X$ open subsets of X, a function $f : V \to k$ is regular when V is considered as an open subset of X if and only if f is regular when V is considered as an open subset of U, and U is considered as a prevariety.

Exercise 2.4. If X is a prevariety, and $\{U_i\}$ is an open cover of X, and we consider each U_i as a prevariety, then for any open subset $V \subseteq X$, and function $f : V \to k$, we have that f is regular if and only if for each i we have $f|_{U_i \cap V}$ is regular when $U_i \cap V$ is considered inside U_i.

Exercise 2.5. If X is a quasiaffine variety, and $U \subseteq X$ open, then $f : U \to k$ is regular in the above sense if and only if it is regular in the sense we defined for quasiaffine varieties.

We then define morphism as usual:

Definition 2.6. Given prevarieties X, Y with atlases given by $\{\varphi_i : X_i \overset{\sim}{\to} U_i\}_i$ and $\{\psi_j : Y_j \overset{\sim}{\to} V_j\}_j$, a **morphism** $\varphi : X \to Y$ is a continuous map such that for all $U \subseteq Y$ open, and all $f : U \to k$ regular, we have $f \circ \varphi : \varphi^{-1}(U) \to k$ is also regular.

It is immediate from Exercise 2.5 that if X, Y are quasiaffine varieties, morphisms $X \to Y$ in the above sense are the same as morphisms in the sense we had already defined. We also see that compositions of morphisms are morphisms. We next see that the condition of being a morphism is a local one.

Exercise 2.7. If $\varphi : X \to Y$ is a morphism, and $U \subseteq X$ is an open subset considered as a prevariety, then $\varphi|_U$ is a morphism.

Conversely, if we have a map $\varphi : X \to Y$ and an open cover U_i of X such that $\varphi|_{U_i}$ is a morphism for each i, then φ is a morphism.

Morphisms are also local on the target.

Exercise 2.8. If $\varphi : X \to Y$ is a morphism, and $V \subseteq Y$ is an open subset of Y, if we consider V and $\varphi^{-1}(V)$ as prevarieties, then φ induces a morphism $\varphi^{-1}(V) \to V$.

Conversely, a continuous map $\varphi : X \to Y$ is a morphism if there is some open cover V_i of Y such that φ induces a morphism $\varphi^{-1}(V_i) \to V_i$ for each i.

A definition of morphism more analogous to a typical one using atlases for differentiable manifolds is then the following:

Exercise 2.9. With notation as in the above definition, a continous map $\varphi : X \to Y$ is a morphism if and only if for any i, j, the induced map

$$(\varphi_i)^{-1}(\varphi^{-1}(V_j)) \overset{\varphi_i}{\to} \varphi^{-1}(V_j) \cap U_i \overset{\varphi}{\to} V_j \overset{(\psi_j)^{-1}}{\to} Y_j$$

is a morphism of quasiaffine varieties.

Following are some basic properties of morphisms.

Exercise 2.10. Prove the following.

(a) If $U \subseteq X$ is open, the inclusion map of prevarieties is a morphism.
(b) If $Z \subseteq X$ is closed and irreducible, the inclusion map of prevarieties is a morphism.
(c) If X and Y are prevarieties, Z a subprevariety of Y, and $\varphi : X \to Y$ any map with $\varphi(X) \subseteq Z$, then φ is a morphism if and only if the induced map $X \to Z$ is a morphism.

Suppose X is a prevariety, and $Y \subseteq \mathbb{A}^n$ an affine variety. Then we see that a function $X \to Y$ is equivalent to an n-tuple of functions $X \to k$, such that the induced map $X \to \mathbb{A}^n$ factors through Y. We can effectively use this correspondence to describe morphisms in terms of n-tuples of regular functions.

Proposition 2.11. *Given a prevariety X and an affine variety $Y \subseteq \mathbb{A}^n$, morphisms $X \to Y$ are equivalent to n-tuples of regular functions on X such that the induced map $\varphi : X \to \mathbb{A}^n$ has image contained in Y.*

In particular, morphisms $X \to Y$ are in bijection with k-algebra homomorphisms $A(Y) \to \mathscr{O}(X)$.

Proof. Certainly, if φ is a morphism, then pulling back the coordinate functions $x_1, \ldots, x_n$ on $\mathbb{A}^n$ gives an n-tuple of regular functions on X, which describe φ. Conversely, suppose the pullbacks of the x_i are regular functions $f_i \in \mathcal{O}(X)$, so we have for any j that $f_i \circ \varphi_j$ is regular on X_j, using our standard atlas notation. But X_j is affine, so this means that the induced map $X_j \to Y$ is a morphism in the classical sense, and (using the one-chart atlas for Y) by Exercise 2.9 we conclude that φ is a morphism.

Now, to prove that morphisms $X \to Y$ are in bijection with k-algebra homomorphisms $A(Y) \to \mathscr{O}(X)$ it is enough to prove that an n-tuple of regular functions $f_1, \ldots, f_n$ on X defines a function which maps X into Y if and only if $g(f_1, \ldots, f_n) = 0$ for all $g \in I(Y)$, and this proceeds just as in the case we already handled, when X is quasiaffine. $\square$

3. Abstract varieties

Note that a prevariety X is never Hausdorff (unless X consists of a single point), since it is irreducible by hypothesis. However, the analogue of the Hausdorff condition for a manifold is precisely what is missing from our definition. It turns out that the right definition involves the following fact from point-set topology:

Exercise 3.1. A topological space X is Hausdorff if and only if the image of the diagonal map $X \to X \times X$ is closed.

We will use the same definitions for varieties, but because the Zariski topology on a product of varieties is not the product topology, we will obtain a different and better-behaved notion.

Of course, we first need to define the product of prevarieties.

Definition 3.2. Given prevarieties X, Y, we define the **product** $X \times Y$ of X with Y to be the product set $X \times Y$, equipped with the atlas

$$\varphi_i \times \varphi_j : X_i \times Y_j \xrightarrow{\sim} U_i \times V_j,$$

and the topology induced by the atlas.

Exercise 3.3. Show the following:

(a) The above definition gives a valid prevariety.
(b) If X and Y are affine, this definition is consistent with the one we already have (and used in the above) for products of affine varieties.
(c) If $Y \subseteq X$ is a subprevariety, then the topology on $Y \times Y \subseteq X \times X$ is the subset topology.

We then have that any prevariety X has a natural diagonal morphism:

Proposition 3.4. *Given a prevariety X, the diagonal map $\Delta : X \to X \times X$ is a morphism of prevarieties.*

Proof. Because it is enough to check locally whether or not a map is a morphism, we can take an atlas $\{\varphi_i : X_i \to U_i\}$ of X and check whether the induced map $U_i \to X \times X$ is a morphism for each i. The image of this map is contained in $U_i \times U_i$, so we reduce to checking whether $U_i \to U_i \times U_i$ is a morphism, and hence we have reduced the proposition to the case that X is affine. But in this case it is clear, as if $X \subseteq \mathbb{A}^n$, the map $X \to X \times X$ is induced by the polynomial map $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, a_1, \ldots, a_n)$. $\square$

Our analogy to the Hausdorff condition is then the following:

Definition 3.5. We say that a prevariety X is a **variety** if the image of the diagonal morphism is closed.

Example 3.6. Any affine variety is a variety. Indeed, if $X = Z(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is an affine variety in $\mathbb{A}^n$, then

$$\Delta(X) = Z(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n), x_1 - x_{n+1}, x_2 - x_{n+2}, \ldots, x_n - x_{2n}),$$

so is closed in $\mathbb{A}^{2n}$.

Example 3.7. An example of a prevariety which is not a variety is given by considering X to be the line with the doubled origin discussed in Example 1.7. We can see explicitly that the diagonal is not closed in this case: if P_1, P_2 denote the two origins, then it is not hard to check that the closure of the diagonal also contains the points (P_1, P_2) and (P_2, P_1), which are not in the diagonal.

Definition 3.8. A **subvariety** of a variety X is a subprevariety.

The terminology is justified by the following.

Proposition 3.9. *Let X be a variety, and $Y \subseteq X$ a subprevariety. Then Y is a variety.*

Proof. We have

$$\Delta(Y) = \Delta(X) \cap (Y \times Y) \subseteq X \times X.$$

Since $\Delta(X)$ is closed and $Y \times Y$ has the subset topology in $X \times X$, we conclude $\Delta(Y)$ is closed in $Y \times Y$. □

The following will be useful for checking that a prevariety is a variety.

Proposition 3.10. *A prevariety X is a variety if and only if for any two points $P, Q \in X$ there is an open subset U of X which contains P and Q and is a variety.*

Proof. If X is a variety, we can take the open subset to be all of X. Conversely, suppose the condition holds; we wish to show $\Delta(X)$ is closed. Thus, suppose (P, Q) is in the closure of $\Delta(X)$. By hypothesis, we can choose U containing P and Q and such that $\Delta(U)$ is closed in $U \times U$. Since $U \times U$ has the subset topology in $X \times X$, and $(P, Q) \in U \times U$, the hypothesis that (P, Q) is in the closure of $\Delta(X)$ implies it is in the closure of $\Delta(U)$, thus in $\Delta(U) \subseteq \Delta(X)$, and since P and Q were arbitrary, we conclude $\Delta(X)$ is closed. □

Example 3.11. The projective line constructed in Example 1.7 is a variety. In order to see this, we claim that if we let V be the open subprevariety obtained by removing the point (1) from each copy of $\mathbb{A}^1_k$ used to define $\mathbb{P}^1_k$, then $V \cong \mathbb{A}^1_k$. This will then prove that $\mathbb{P}^1_k$ is a variety using Proposition 3.10, since any two points are contained either in one of the two copies of $\mathbb{A}^1_k$ in the atlas for $\mathbb{P}^1_k$, or in V.

Now, V is defined by gluing two copies of $Y = \mathbb{A}^1_k \smallsetminus (0)$ to each other via the map $t \mapsto 1/t$. Denote the two copies of Y by V_1 and V_2. Recalling that Y is affine, isomorphic to $Z(x, y) \subseteq \mathbb{A}^2_k$, we see that the V_i give an atlas for V as a prevariety. To construct an isomorphism to $\mathbb{A}^1_k$, consider the regular function defined by $1/(t-1)$ on V_1, and by $1/(1/t - 1) = t/(1-t)$ on V_2. This defines a morphism to $\mathbb{A}^1_k$, with inverse morphism defined by sending x to $(x+1)/x$ in V_1 for $x \neq 0$, and to $x/(x+1)$ in V_2 for $x \neq -1$.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.