Exercise 1. (a) Show that if \( f \in A_n \) is a nonzero polynomial, then \( \mathbb{Z}(f) \neq \mathbb{A}^n \).

(b) Without using the ideal-variety correspondence, show that \( \mathbb{A}^n \) is irreducible for all \( n \).

Solution. (a) I intended a direct argument for this, but since I didn’t explicitly say so, I allowed arguments involving ideals. The argument I had in mind is the following: induct on \( n \), using that for \( n = 1 \) we know that a nonzero polynomial has finitely many roots, and \( k \) is infinite. Then if \( f \in A_n \) is nonzero, consider \( f \) as a nonzero polynomial in \( x^n \) with coefficients in \( A_{n-1} \); by assumption, at least one coefficient is nonzero, so by applying the induction hypothesis to that coefficient, there exists \((c_1, \ldots, c_{n-1})\) such that \( f(c_1, \ldots, c_{n-1}, x) \) is nonzero in \( k[x_n] \). Applying the \( n = 1 \) case again we can find \( c_n \) such that \( f(c_1, \ldots, c_n) \neq 0 \), as desired.

(b) Suppose \( T_1, T_2 \subseteq A_n \) are both (nonempty and) not equal to \( (0) \). Then choose \( f_i \in T_i \) nonzero for \( i = 1, 2 \), and note that by definition
\[
\mathbb{Z}(T_1) \cup \mathbb{Z}(T_2) \subseteq \mathbb{Z}(f_1) \cup \mathbb{Z}(f_2) = \mathbb{Z}(f_1f_2) \subsetneq \mathbb{A}^n
\]
by (a) and the fact that \( A_n \) is an integral domain. It follows that \( \mathbb{A}^n \) is irreducible.

Exercise 2. Show that the Zariski topology on \( \mathbb{A}_k^2 \) is not equal to the product topology, identifying \( \mathbb{A}_k^2 \cong \mathbb{A}_k^1 \times \mathbb{A}_k^1 \).

Solution. Most people did fine on this, describing either the open or closed subsets in the product topology, and going from there. Note that the infinitude of the base field should arise, since the Zariski topology and the product topology on \( \mathbb{A}_k^2 \) are in fact the same over a finite field.

One can also show directly that the diagonal \( x = y \) isn’t closed in the product topology, without giving a complete description of the closed sets, by showing that its complement doesn’t contain any product of nonempty open subsets of \( \mathbb{A}_k^1 \). This amounts to proving or invoking the fact from topology that a space is Hausdorff if and only if its diagonal is closed.

Exercise 3. Consider \( \mathbb{Z}(x^2 - yz, xz - x) \subseteq \mathbb{A}_k^3 \). Find the irreducible components, and the corresponding prime ideals of \( k[x, y, z] \).

Solution. Most people did fine with this, but I wanted to see at least a brief justification of why \( \mathbb{Z}(x^2 - yz, xz - x) = \mathbb{Z}(x, y) \cup \mathbb{Z}(x, z) \cup \mathbb{Z}(x^2 - y, z - 1) \), and more importantly, why the corresponding ideals are prime (since that’s how you know you actually found the irreducible components). For the latter, it is easiest to argue by describing the quotient rings as polynomial rings in one variable. I didn’t enforce this in the grading, but this could be a bit tricky. For instance, we have homomorphisms \( k[x] \to k[x, y, z]/(x^2 - y, z - 1) \) and \( k[x, y, z]/(x^2 - y, z - 1) \to k[x] \), but it’s not completely obvious that they are inverse to one another: one composition is obviously the identity, but checking that the other is the identity amounts to checking directly the injectivity of the second map.

Exercise 4. All of our definitions so far make perfect sense even if \( k \) is not algebraically closed, and most of our basic results still hold. In fact, the only algebra theorem we have used which requires algebraic closure is the Nullstellensatz.
(a) Over your favorite non-algebraically closed field, give a counterexample to the Nullstellensatz.

(b) Give an example of a non-algebraically closed field \( k \) and an irreducible polynomial \( f \in k[x, y] \) such that \( Z(f) \) is nonempty but not irreducible (hint: there are various ways to do this, but one approach is to find an irreducible complex polynomial in two variables which is not a scalar times a real polynomial, and whose real zeroes consist of more than one (but finitely many) points).

Solution. No significant problems with this. For (b), the hint has in mind to find the polynomial in question and multiply it by its complex conjugate. One can also just take a logic-style approach, encoding multiple vanishing conditions over \( \mathbb{R} \) by taking sums of squares to arrive at an example like \( y^2 + x^2(x - 1)^2 \) (but don’t forget to check irreducibility). It’s also easy to work over finite fields, where every zero set with more than one point is automatically reducible.