Exercise 1. Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set, with $f_1, \ldots, f_m$ generating $I(X)$. Then given a point $P \in X$, and a vector $v \in k^n$, the following are equivalent:

(a) $v$ is a tangent vector to $X$ at $P$;
(b) we have
$$ (\partial f_i/\partial x_1(P), \ldots, \partial f_i/\partial x_n(P)) \cdot v = 0 $$
for $i = 1, \ldots, m$;
(c) we have
$$ (\partial f/\partial x_1(P), \ldots, \partial f/\partial x_n(P)) \cdot v = 0 $$
for all $f \in I(X)$.

Solution. Generally there weren’t a lot of problems with this. One can avoid redundancy by first proving that if $f \in I(X)$, then the linear coefficient of $f(tv+P)$ is equal to $(\partial f/\partial x_1(P), \ldots, \partial f/\partial x_n(P)) \cdot v$, which immediately gives the equivalence of (a) and (c), and reduces the problem to showing that (b) implies (c).

It’s fine to use things like the multivariable chain rule (which can actually be deduced from the real case, although that’s a bit perverse), but one can also just argue directly from the definitions without much extra work. However, a note of caution: for a polynomial $f(t)$ with $f(0) = 0$, it is always true that $f$ vanishes to order at least 2 if and only if $f'(0) = 0$. But this fails in positive characteristic for higher orders of vanishing!

Exercise 2. Do Exercise 4.7 of Chapter I of Hartshorne.

Solution. This was probably the most difficult problem. I believe it is possible to deduce this statement from the corresponding birational statement on field isomorphisms (Theorem 3.3.9 and Corollary 3.3.11 of the notes), but this is rather subtle. It is more straightforward to mimic the proof of these results. The proof can be phrased in various ways; what I have done below follows the proof of 3.3.9 rather closely.

Two points to be careful of: an arbitrary homomorphism of local rings need not map the maximal ideal to the maximal ideal, so here we are using that we are starting with an isomorphism. Also, the ideal-point correspondence only holds for affine varieties, so you should be careful to consider only neighborhoods isomorphic to affine varieties if you are invoking it.

First, since the problem is local around $P$ and $Q$, we may assume that $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are affine (either by taking closures in affine space if we assume $X$ and $Y$ are quasiaffine, or more generally by taking open neighborhoods of $P$ and $Q$ isomorphic to affine varieties). Then we have $\mathcal{O}_{P,X} = A(X)_{m_P}$ and $\mathcal{O}_{Q,Y} = A(Y)_{m_Q}$, so an isomorphism $\varphi_Q^* : \mathcal{O}_{Q,Y} \sim \mathcal{O}_{P,X}$ can be considered as an isomorphism $A(Y)_{m_Q} \sim A(X)_{m_P}$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be the ambient variables for $X$ and $Y$ respectively, so that they also are generators for $A(X)$ and $A(Y)$. For $i = 1, \ldots, m$, write $\varphi_Q^*(y_i) = \frac{g_i}{h_i}$ with $g_i, h_i \in A(X)$, and $h_i \notin m_P$. Let $h = \prod_i h_i$; then $h \notin m_P$ also, so if we let $X' = X \setminus Z(h)$, we still have $P \in X'$. Now $X'$ is still (isomorphic to) an affine variety, with $A(X') = A(X)_h$, so we see that $\varphi_Q^*$ induces a homomorphism $A(Y) \to A(X')$, which thus comes from a morphism $\varphi : X' \to Y$. 

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We next claim that this morphism sends \( P \) to \( Q \); one can state this in terms of the ideal-point correspondence, or we can see it directly as follows: for any regular function \( f \) on \( Y \), if \( f(Q) = 0 \), then \( f \in \mathfrak{m}_Q \), and since \( \varphi^* \) was an isomorphism of local rings, it necessarily sent \( \mathfrak{m}_Q \) to \( \mathfrak{m}_P \), so \( f \circ \varphi = \varphi^*(f) = \varphi_Q^*(f) \in \mathfrak{m}_P \). Thus, \( f(\varphi(P)) = 0 \). But for any \( Q' \neq Q \in Y \), we can find \( f \in A(Y) \) vanishing at \( Q \) and not vanishing at \( Q' \), so we conclude that \( \varphi(P) = Q \), as desired.

Next, we apply the same construction in the other direction, switching the roles of \( X \) and \( Y \) and putting \( X' \) in place of \( X \), and we obtain \( Y' = Y \setminus Z(h') \) and a morphism \( \psi : Y' \to X' \) sending \( Q \) to \( P \), which is a birational inverse to \( \varphi \). Finally, if we set

\[
X'' = X' \setminus Z(\varphi^*(h')) = X' \setminus \varphi^{-1}(Z(h')),
\]

we note that \( P \in X'' \) since \( \varphi(P) = Q \) and \( Q \notin Z(h') \), and we also see that \( \varphi \) induces a morphism \( X'' \to Y' \). We claim that this is an isomorphism. Since \( X'' \) and \( Y' \) are affine, this is probably easiest to check on the level of rings: we have that \( \varphi^* \) and \( \psi^* \) are inverse on the level of function fields, so it suffices to see that they map the subrings \( A(X'') \) and \( A(Y') \) into one another. By construction, \( \varphi^* \) maps \( A(Y') \) into \( A(X'') \), and \( \psi^* \) maps \( A(X'') \) into \( A(Y') \). But \( \psi^*(\varphi^*(h')) = h' \), so since \( h' \) is invertible in \( A(Y') \), we see that \( \psi^* \) also maps \( A(X'') = A(X') \varphi^*(h') \) into \( A(Y') \), and we conclude the desired isomorphism.

**Exercise 3.** Do Exercise 5.2 of Chapter I of Hartshorne.

**Solution.** This was also mostly straightforward. Some people didn’t notice they had to check that the ideals are generated by the polynomials in question (but see below), or did notice it but didn’t justify why the polynomials are irreducible (not entirely trivial in the last case). For the first two parts, it happens that the set of points where the Jacobian is zero all lie on the surface, but don’t forget that this has to be checked as well. For the last part, the simplest solution involves a little trick: we want the common zeros of \( xy + x^3 + y^3 \), \( y + 3x^2 \), \( x + 3y^2 \). Multiplying the second and third by \( x \) and \( y \) respectively, adding them together, and subtracting three times the first gives \( -xy = 0 \). If you don’t do this, be careful – the intersection of \( y + 3x^2 = 0 \) and \( x + 3y^2 = 0 \) has more than two possibilities for \((x, y)\)!

There is actually a trick one can use to avoid checking that the polynomials are irreducible. In fact, we just need to know that they generate the ideal, which means that they don’t have any factors appearing with exponent bigger than one. But if there were a multiple factor, the Jacobian would vanish along its zero set, so you would be seeing 2-dimensional “singularities.” Since you only found 1-dimensional (or 0-dimensional) ones, we can actually conclude from the Jacobian calculation that the polynomials generate the ideals.

**Exercise 4.** Let \( Y \) be the \( 2 \times 3 \) matrices of rank at most 1.

(a) Find the singular points of \( Y \).

(b) For each nonsingular point \( P \) of \( Y \), give a pair of polynomials which suffices to define \( Y \) locally near \( P \).

**Solution.** (a) We still don’t know that \( I(Y) \) is generated by the \( 2 \times 2 \) minors \( m_1 = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}, m_2 = x_{1,1}x_{2,3} - x_{1,3}x_{2,1}, m_3 = x_{1,2}x_{2,3} - x_{1,3}x_{2,2} \), but we do at least know they are in \( I(Y) \), so we can apply one direction of the Jacobian criterion. Since \( Y \) has codimension 2, the points \( P \) where \( J(m_1, m_2, m_3) \) has rank equal to 2 are nonsingular. But the \( 2 \times 2 \) minors of \( J(m_1, m_2, m_3) \) include among them the squares of each of the \( x_{i,j} \), so we conclude that \( J(m_1, m_2, m_3) \) has rank 2 everywhere except at the zero matrix, so \( Y \) is nonsingular everywhere except possibly at the zero matrix.

To see that the zero matrix is indeed a singular point of \( Y \), recall that we have already shown that \( I(Y) \) can’t contain any polynomials with nonzero linear terms. It then follows that whatever polynomials are in \( I(Y) \), all their partial derivatives necessary vanish at the zero matrix, so we can
conclude that the zero matrix is in fact a singular point of $Y$. Alternatively, one can show directly that the Zariski cotangent space of $Y$ at the zero matrix is the same as the Zariski cotangent space of $A^6_k$, again using that $I(Y)$ is contained in the square of the maximal ideal.

(b) According to lecture, it suffices to give, for each $P$, a pair of the $m_i$ whose corresponding rows of the Jacobian are linearly independent. We see by inspection that $m_1, m_2$ will define $Y$ locally if $x_{11}$ or $x_{21}$ is nonzero; $m_1, m_3$ will define $Y$ locally if $x_{2,2}$ or $x_{1,2}$ is nonzero; and $m_2, m_3$ will define $Y$ locally if $x_{2,3}$ or $x_{1,3}$ is nonzero. One can express this more succinctly as saying that if a given $x_{i,j}$ is nonzero, then $Y$ is defined locally by the two minors in which $x_{i,j}$ appears. (As an alternative to the corollary of the Jacobian criterion, one can show this directly: for instance, if $x_{1,1} \neq 0$ and $m_1 = m_2 = 0$, then $x_{2,2} = \frac{x_{1,2}x_{2,1}}{x_{1,1}}$ and $x_{2,3} = \frac{x_{1,3}x_{2,1}}{x_{1,1}}$, so $m_3 = 0$ necessarily.)