We will show that every nonsingular complex variety is a manifold in the analytic topology, but we wish to show more – that it has the natural structure of a complex manifold. In order to do this, we will need to introduce additional structure on the analytic topology. To allow for singular points, we give a definition more general than that of complex manifolds, mimicking our definitions from affine varieties and abstract varieties. With this additional structure, we will conversely be able to test nonsingularity using the analytic structure.

1. Analytic functions and analytic spaces

First recall that a function on an open subset of $\mathbb{C}^n$ is analytic if it can be represented in an open neighborhood of every point by a power series expansion. We then begin with affine definitions, using what is probably nonstandard terminology.

Definition 1.1. An affine analytic space is a subset $X \subset \mathbb{C}^n$ such that in a neighborhood of every point of $X$, there exist finitely many analytic functions with common zero set equal to $X$.

Given an affine analytic space $X \subseteq \mathbb{C}^n$, and an open subset $U \subseteq X$ (in the topology induced by the usual analytic topology on $\mathbb{C}^n$), a function $f: U \to \mathbb{C}$ is analytic if for every point $P \in U$, there is a neighborhood $V \subseteq \mathbb{C}^n$ of $P$ and an analytic function $\tilde{f}: V \to \mathbb{C}$ such that $\tilde{f}|_{U \cap V} = f|_{U \cap V}$.

Note that because the definition is local, an open subset of an affine analytic space is still an affine analytic space.

Example 1.2. Since rational functions are analytic on their domain, a complex quasiaffine variety has the structure of an affine analytic space, and the analytic topology is compatible with this definition. Moreover, regular functions yield analytic functions.

We can then define general analytic spaces using atlases.

Definition 1.3. An analytic space $X$ is a topological space, together with a countable open cover $\{U_i\}$, and a collection of homeomorphisms $\varphi_i : X_i \to U_i$, where each $X_i \subset \mathbb{C}^{n_i}$ is an affine analytic space, and we require that every transition map

\[ \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j) \]

is analytic. We say that each map $\varphi_i : X_i \to U_i \subseteq X$ is a chart, and the collection of charts is an atlas.

In particular, a complex prevariety $X$ has a natural structure of an analytic space, with the topology $X_{an}$ corresponding to the topology on the associated analytic space. From now on, we use $X_{an}$ to denote this associated analytic space structure on $X$.

We can then define complex manifolds as a special case of analytic spaces:

Definition 1.4. A complex manifold $X$ is a Hausdorff analytic space such that each $X_i$ in the atlas is in fact an open subset of $\mathbb{C}^{n_i}$.
We now define analytic functions and analytic maps of analytic spaces.

**Definition 1.5.** If $X$ is an analytic space with a given atlas, and $U \subseteq X$ is open, a function $f : U \to \mathbb{C}$ is **analytic** if for all $i$, the induced function

$$f \circ \varphi_i : \varphi_i^{-1}(U \cap U_i) \to \mathbb{C}$$

is analytic (in the sense defined for affine analytic spaces).

**Definition 1.6.** Given analytic spaces $X$, $Y$, an **analytic map** $\varphi : X \to Y$ is a continuous map such that for all $U \subseteq Y$ open, and all analytic functions $f : U \to \mathbb{C}$, we have $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{C}$ is also an analytic function.

**Exercise 1.7.** With notation as in the above definition, a continuous map $\varphi : X \to Y$ is analytic if and only if for all $i, j$, the induced map

$$(\varphi_i)^{-1}(\varphi^{-1}(V_j)) \xrightarrow{\varphi_i} \varphi^{-1}(V_j) \cap U_i \xrightarrow{\varphi_i} V_j \xrightarrow{(\psi_j)^{-1}} Y_j$$

is an analytic map of affine analytic spaces.

**Example 1.8.** A morphism of complex prevarieties gives an analytic map of the associated analytic spaces. Indeed, this follows from the fact that on an affine open cover, the morphism is given by tuples of polynomials, which are thus analytic.

Abusing terminology slightly, we will say an analytic space is a complex manifold if it is isomorphic to one.

**Remark 1.9.** If $X$ is a complex manifold, then it is naturally oriented. Indeed, $\mathbb{C}^n$ carries a natural orientation which is inherited by its open subsets, and because the transition maps are analytic, they preserve orientation, giving a well-defined orientation on all of $X$.

**Remark 1.10.** It is visibly false that every affine analytic space is algebraic; for instance, the set $y = e^x$ is not algebraic. Remarkably, Chow’s theorem states that if $X$ is a closed analytic subspace on $\mathbb{P}^n_{\mathbb{C}}$, then $X$ is algebraic. A strengthened form of this result also implies that $X$ is determined uniquely by $X_{\text{an}}$, so it makes sense to talk about whether or not a given analytic space “is algebraic”. Chow’s theorem also implies that every compact Riemann surface is algebraic.

**Remark 1.11.** Finally, we remark that the definition of local ring carries over to analytic spaces, with the same slightly modified definition that we used for algebraic sets (the reason we need this is that the topology is finer, so for instance there are generally many disjoint open sets). Similarly, an analytic map induces homomorphisms on the local rings in the opposite direction, just as in the case of morphisms of varieties.

### 2. Nonsingular Points and Complex Manifolds

We can now prove the following theorem:

**Theorem 2.1.** Let $X$ be a complex prevariety of dimension $d$. Then $X$ is a nonsingular variety if and only if $X_{\text{an}}$ is a complex manifold, and in this case, $X_{\text{an}}$ also has dimension $d$.

**Proof.** We first recall that we have already proved that $X_{\text{an}}$ is Hausdorff if and only if $X$ is a variety, so we are reduced to proving a local statement, and can assume that $X$ is affine.

Suppose $X$ is nonsingular. Let $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ be a set of polynomials generating $I(X)$. According to the Jacobian criterion, the nonsingularity of $X$ is equivalent to the matrix

$$(\frac{\partial f_i}{\partial x_j})_{i,j}$$

having rank $n - d$ at all points of $X$. In particular, for any point $P \in X$ there is some $(n-d) \times (n-d)$ minor which is nonvanishing at $P$. Without loss of generality, suppose that the $f_i$ are ordered so that $(\frac{\partial f_i}{\partial x_j})_{1 \leq i \leq n-d}$ has rank $n - d$ at $P$. Fixing a nonvanishing minor, let $J \subseteq \{1, \ldots, n\}$
be the $d$ indices not involved in the minor, so that $\det(\frac{\partial f_j}{\partial x_i})_{1 \leq i \leq n - d, j \notin J}$ is non-vanishing at $P$. By our earlier theorem, we know that in a neighborhood of $\tilde{P}$, we have $X = Z(f_1, \ldots, f_{n-d})$. It then follows from the implicit function theorem that the projection morphism onto the coordinates $x_j$ for $j \in J$ has analytic inverse in a neighborhood of the image of $P$, so we conclude that a neighborhood of $P$ is analytically isomorphic to an open subset of $\mathbb{C}^d$, and thus that $X$ is a complex manifold of dimension $d$.

We leave the converse statement as Exercise 2.2. \hfill \Box

**Exercise 2.2.** Suppose $X$ is a complex affine variety of dimension $d$, and $P \in X$ any point.

a) Show that the Zariski cotangent space $T^*_P(X)$ is isomorphic to $\mathfrak{m}_{an,P}/\mathfrak{m}_{an,P}^2$, where $\mathfrak{m}_{an,P}$ is the maximal ideal of the local ring of $X_{an}$ at $P$.

b) Conclude that if an open neighborhood $U$ of $P$ in $X_{an}$ is a complex manifold of dimension $d$, then $P$ is a nonsingular point of $X$.

c) Using that the nonsingular points of any variety form a non-empty Zariski open subset, conclude that if $X_{an}$ is a complex manifold, then $X$ is nonsingular.

### 3. Applications of complex techniques

We have already seen one application of analytic techniques, on the first day of class: for elliptic curves over the complex numbers, the subgroup of $n$-torsion points is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. As another application, we have that if $X$ is a projective nonsingular complex curve, then $X_{an}$ is a compact Riemann surface, so we can define its genus as the topological genus of the underlying compact orientable surface. This agrees with our earlier definition, but gives a more geometric point of view.

More substantively, tools from topology applied to the analytic structure on complex varieties have played a vital role in much of algebraic geometry, both historically and in modern research. They have also motivated many of the important technical tools developed in the second half of the 20th century to apply classical topological ideas to varieties over arbitrary fields.

One class of applications of analytic techniques to complex varieties involves using these techniques to prove cohomological statements on cohomology, such as vanishing theorems or other results. These can then often be applied to other questions, such as the study of Jacobians. An example of this is the exponential exact sequence, discussed in §B.5 of Hartshorne [1].

We mention two other applications. One is due to Brieskorn, and is an application to differential topology. Namely, Brieskorn showed that if one takes the intersection of a small sphere about the origin with the complex affine variety

$$x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_5^{6k-1} = 0$$

for $k = 1, \ldots, 28$, one always obtains a topological space homeomorphic to the sphere $S^7$, but the associated analytic space induces differential structures which are different from one another, and indeed these give all 28 possible differential structures on $S^7$.

Finally, we discuss an application to rationality. Luroth proved that any field lying between $k$ and the transcendental extension $k(x)$ (here $k$ is algebraically closed) is itself isomorphic to $k(x)$. Geometrically, this says that if $X$ is a variety over $k$, and there is a dominant rational map $\mathbb{P}^1_k \dashrightarrow X$, then $X$ is rational (i.e., is birational to $\mathbb{P}^1_k$). We have observed that this follows from the Riemann-Hurwitz formula, which implies that genus cannot go down under a dominant morphism of projective nonsingular curves. More generally, if $X$ is a variety and there exists a dominant rational map $\mathbb{P}^n_k \rightarrow X$ for some $n$, we say $X$ is **unirational**. The Luroth problem then asks whether it is true that every unirational variety is rational, or equivalently, whether a subfield of $k(x_1, \ldots, x_n)$ strictly containing $k$ is purely transcendental. We have answered this question affirmatively in dimension 1. In dimension 2, it remains true under a separability hypothesis (and
in particular in characteristic 0), and is a theorem of Castelnuovo, following from the classification of surfaces. See Remark V.6.2.1 of Hartshorne [1].

For a long time, it was open whether unirationality implies rationality in dimensions higher than 2. This was finally settled negatively in the early 1970’s in papers by Iskovskih and Manin, by Artin and Mumford, and by Clemens and Griffiths. We briefly discuss the work of Clemens and Griffiths. In their influential paper, they worked with a very explicit class of examples: 3-dimensional varieties defined by a cubic polynomial in \( \mathbb{P}^4 \). These had been known to be unirational, and they proved that they are not rational. Their techniques were very analytic, using Hodge theory and a generalization of the Jacobian to prove that one cannot have birationality between the cubic threefold and \( \mathbb{P}^3_{ \mathbb{C} } \).

References