SECANT VARIETES AND CURVES IN PROJECTIVE SPACE

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We have seen that every nonsingular curve is quasiprojective. We make a preliminary elaboration of this by studying what kind of morphisms we can construct into specific projective spaces. Specifically, we will prove:

Proposition 1. *Suppose C is a nonsingular curve. Then there exists an injective morphism $\varphi : C \to \mathbb{P}^3$. Given any $Q \in C$, there also exists a morphism $\varphi' : C \to \mathbb{P}^2$ such that*

$$(\varphi')^{-1}(\varphi'(Q)) = \{Q\}.$$

This statement will be useful in analyzing the topology of complex varieties, and it follows immediately from the following two more general results.

Proposition 2. *Suppose X is a quasiprojective variety of dimension d. Then there exists an injective morphism $\varphi : X \to \mathbb{P}^{2d+1}$.*

Proposition 3. *Given $Q \in X$ any point on a quasiprojective variety of dimension d, there exists a morphism $\varphi' : X \to \mathbb{P}^{d+1}$ such that $(\varphi')^{-1}(\varphi(Q)) = \{Q\}$.*

The idea of the argument is that we start with a realization of X as a subvariety of some $\mathbb{P}^n$, and then show we can project inductively to smaller-dimensional projective spaces. The key construction is:

Definition 4. Given a subvariety $X \subseteq \mathbb{P}^n$, the **secant variety** $\operatorname{Sec}(X)$ of X is the Zariski closure of the set of points P of $\mathbb{P}^n$ such that there exist distinct points $Q_1, Q_2 \in X$ with P lying on the line through Q_1, Q_2.

Proposition 5. *The secant variety of X is a variety, of dimension less than or equal to $2 \dim X + 1$.*

Remark 6. Sometimes, $\operatorname{Sec}(X)$ is defined without taking the Zariski closure, but then it requires more care to check that it is in fact a subvariety.

Proof. Let d be the dimension of X. We first consider the auxiliary variety $\widetilde{\operatorname{Sec}}(X) \subseteq \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ defined by triples of points (Q_1, Q_2, P) with $Q_1, Q_2 \in X$ distinct, and P on the line between Q_1 and Q_2. By definition, $\operatorname{Sec}(X)$ is the closure of the image of $\widetilde{\operatorname{Sec}}(X)$ under the third projection morphism. It thus suffices to prove that $\widetilde{\operatorname{Sec}}(X)$ is a variety of dimension equal to $2d + 1$.

We first see that it is an open subset of a closed subset of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. Indeed, it is the intersection of $X \times X \times \mathbb{P}^n$ with the set of triples of points $(Q_1, Q_2, P) \in (\mathbb{P}^n \times \mathbb{P}^n \smallsetminus \Delta(\mathbb{P}^n)) \times \mathbb{P}^n)$ such that P lies on the line between Q_1 and Q_2. The latter set can be described by the condition that the coordinate vectors of Q_1, Q_2 and P are linearly dependent, which is a polynomial condition in the coordinates of the three points given by the vanishing of the 3×3 minors of the associated $3 \times (n + 1)$ matrix (the matrix is defined only up to nonzero scaling of each row, but this will not affect whether or not a given minor vanishes). Thus $\widetilde{\operatorname{Sec}}(X)$ is an open subset of a closed subset, and it suffices to see that it is irreducible of dimension $2d + 1$.

We prove both statements at once as follows: let $U_0, \dots, U_{n+1}$ be the affine open cover of X with $U_i = X \smallsetminus Z(x_i)$. For any i, j, let $U_{i.j} \subseteq \widetilde{\operatorname{Sec}}(X)$ be the preimage of $U_i \times U_j$ under the projection morphism to the first two factors; that is,

$$U_{i,j} = \{(Q_1, Q_2, P) \in \widetilde{\operatorname{Sec}}(X) : Q_1 \in U_i, Q_2 \in U_j\}.$$

It is clear that the $U_{i,j}$ form an open cover of $\widetilde{\mathrm{Sec}}(X)$. We claim that $U_{i,j} \cong U_i \times U_j \times \mathbb{P}^1$. For each U_i, we normalize coordinates in $\mathbb{P}^n$ so that $x_i = 1$. We then obtain a morphism $\psi_{i,j} : U_i \times U_j \times \mathbb{P}^1 \to U_{i,j}$ by sending $((x_0, \dots, x_n), (y_0, \dots, y_n), (s, t))$ to $(x_0, \dots, x_n), (y_0, \dots, y_n), (sx_0 + ty_0, \dots, sx_n + ty_n)$. Note that $\psi_{i,j}$ is well defined because we have normalized coordinates on U_i, U_j and simultaneously scaling (s, t) will scale the last $(n + 1)$-tuple. Furthermore, $\psi_{i,j}$ is visibly bijective, so we wish to show that the inverse is a morphism. In order to do so, we may restrict to the open cover $V_{a,b}$ of $U_{i,j}$ consisting of points $((x_0, \dots, x_n), (y_0, \dots, y_n), (z_0, \dots, z_n))$ on which (x_a, x_b) is linearly independent from (y_a, y_b). Recall that $(x_0, \dots, x_n) \neq (y_0, \dots, y_n)$, so this does in fact form a cover. On $V_{a,b}$, we see that $\psi_{i,j}^{-1}$ is expressed by setting

$$s = y_b z_a - y_a z_b, t = x_a z_b - x_b z_a;$$

after composition, (s, t) is scaled by $x_a y_b - x_b y_a$, which doesn't change the point in $\mathbb{P}^1$. Thus $\psi_{i,j}^{-1}$ is also a morphism, and $\psi_{i,j}$ is an isomorphism.

Since $U_i \times U_j \times \mathbb{P}^1$ is a variety of dimension $2d + 1$, we conclude the same for $U_{i,j}$. Since $\widetilde{\mathrm{Sec}}(X)$ is covered by the $U_{i,j}$, the only point that remains is to check irreducibility. We observe that given i', j', we have $U_i \cap U_{i'} \neq \emptyset$ and $U_j \cap U_{j'} \neq \emptyset$, so $U_{i,j} \cap U_{i',j'} \neq \emptyset$. Since the $U_{i,j}$ form an open cover, and each is irreducible, we conclude that $\widetilde{\mathrm{Sec}}(X)$ does not contain any pair of disjoint non-empty open subsets, and thus is irreducible, as desired. □

We can now prove the main propositions.

Proof of Proposition 2. By hypothesis, we can realize $X \subseteq \mathbb{P}^n$ for some n. We prove by downward induction that there is an injective morphism $\varphi : X \hookrightarrow \mathbb{P}^{2d+1}$. It suffices to show that given an injective morphism $\varphi_m : X \to \mathbb{P}^m$, if $m > 2d + 1$ we can produce an injective morphism $\varphi_{m-1} : C \to \mathbb{P}^{m-1}$. Let X_m be the Zariski closure of $\varphi_m(X)$; then it is also a variety of dimension d. Now choose $P \in \mathbb{P}^m \smallsetminus \mathrm{Sec}(X_m)$, which is nonempty since $m > 2d+1$ and $\dim \mathrm{Sec}(X_m) \leqslant 2d+1$ by Proposition 5. Let H be any hyperplane not containing P, and let $\varphi_{m-1} = \pi_P \circ \varphi_m$, where $\pi_P : \mathbb{P}^m \smallsetminus \{P\} \to H$ is the projection morphism sending a point Q to the point of H lying on the line through P and Q. Note that $X_m \subseteq \mathrm{Sec}(X_m)$, so π_P is defined on X_m. Since φ_m is injective by hypothesis, it suffices to see that $\pi_P|_{X_m}$ is injective. But two points $Q_1, Q_2 \in X_m$ have $\pi_P(Q_1) = \pi_P(Q_2)$ if and only if the line through P and Q_1 is the same as the line through P and Q_2, which can only happen if $Q_1 = Q_2$ by the hypothesis that $P \notin \mathrm{Sec}(X_m)$. Thus we have produced the desired φ_{m-1}. □

Proof of Proposition 3. We mimic the argument of Proposition 2. Suppose that $\varphi_m : X \to \mathbb{P}^m$ is a morphism such that $(\varphi_m)^{-1}(\varphi_m(Q)) = \{Q\}$, and $m > d + 1$ (as before, we can start with any imbedding of X in some $\mathbb{P}^n$). Arguing as in Proposition 5, it is clear that for any $Y \subseteq \mathbb{P}^m$ of dimension d, and $Q_1 \in Y$, the set of points $P \in \mathbb{P}^m$ such that P lies on the line connecting Q_1 to some $Q_2 \neq Q_1$ on X has dimension at most $d + 1$. Thus if we choose P not in this set, and H any hyperplane not containing P, and we set $\varphi_{m-1} = \pi_P \circ \varphi_m$, we find that φ_{m-1} satisfies $(\varphi_{m-1})^{-1}(\varphi_{m-1}(Q)) = \{Q\}$. Inducting downwards, we obtain the desired φ'. □

Remark 7. The nonsingularity hypothesis in Proposition 1 is in fact unnecessary; we include it only because we have not proved that arbitrary curves are quasiprojective. However, an elaboration of the techniques used above allows one to prove that every nonsingular curve is isomorphic to a curve in $\mathbb{P}^3$, and birational to a nodal curve in $\mathbb{P}^2$. These statements do require nonsingularity. See Chapter IV, Corollary 3.6 and Theorem 3.10 of [1].

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.