POWER SERIES AND NONSINGULAR POINTS

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One of the fundamental facts about a nonsingular point of an affine variety X is that in a neighborhood of the point, the ideal of X can be generated by the "expected" number of polynomial equations. In order to prove this, we will study more generally nonsingular points of algebraic sets, proving that there cannot be multiple irreducible components meeting at a nonsingular point. Our main tool will be power series expansions near the point in question.

We begin by remarking that nearly all our definitions for (affine) varieties make sense also for algebraic sets, which is to say without the irreducibility hypothesis. The biggest difference is that the function field no longer makes sense. We also have to make a slight modification to the equivalence relation in the definition of the local ring at a point P: if (U, f) and (V, g) are pairs with U, V open neighborhoods of P and f, g regular on U, V respectively, then $(U, f) \sim (V, g)$ if there exists an open neighborhood W of P contained in $U \cap V$ such that $f|_W = g|_W$. This then gives an equivalence relation for any algebraic set, and agrees with our old definition in the case of a variety. The definition of the Zariski cotangent space is identical, and we modify the definition of nonsingular point very slightly: a point $P \in X$ is nonsingular if $\dim_k T^*_P(X) = \dim_P X$, where $\dim_P X = \max_Z \dim Z$, where Z ranges over components of X containing P.

The main point we will use is then that the argument for the Jacobian criterion for nonsingularity did not use anywhere the hypothesis that the variety was irreducible, and thus applies also to algebraic sets.

1. Completions

We begin by discussing completions of local rings, which play a role in algebraic geometry closely analogous to that of passing to a small neighborhood in differential or complex geometry.

Definition 1.1. Let R be a local ring, with maximal ideal $\mathfrak{m}$. Then the **completion** $\hat{R}$ of R is the inverse limit $\varprojlim_n R/\mathfrak{m}^n$.

This means that elements of $\hat{R}$ correspond to infinite sequences $r_1, r_2, \ldots$ with $r_n \in R/\mathfrak{m}^n$, and such that for each $n > 1$, we have that $r_{n_1} \equiv r_n \pmod{\mathfrak{m}^{n-1}}$. This is a ring, since addition and multiplication can be defined on individual terms of the sequence. Given $r \in R$, we get an element of $\hat{R}$ simply by taking the sequence $r, r, \ldots$. We easily conclude:

Proposition 1.2. *For any local ring R, the map $r \mapsto r, r, \ldots$ gives a canonical homomorphism $R \to \hat{R}$.*

A basic fact of algebra states:

Theorem 1.3 (Krull intersection)**.** *If R is a Noetherian local ring, then the canonical homomorphism $R \to \hat{R}$ is injective.*

In particular, if P is a point of an algebraic set X, then the local ring $\mathcal{O}_{P,X}$ injects into its completion $\hat{\mathcal{O}}_{P,X}$. This injection plays a role analogous to Taylor series expansion, although the Krull intersection theorem says that unlike the real differentiable situation, in algebraic geometry there is no non-zero function whose Taylor series is 0. We next make the analogy with Taylor series more precise.

2. Power series and the completion

Given formal variables $x_1, \ldots, x_r$, recall that the formal power series ring $k[[x_1, \ldots, x_r]]$ has elements consisting of infinite formal sums

$$\sum_{i=0}^{\infty} F_i,$$

where each F_i is a homogeneous polynomial of degree i in the x_i. These form a (commutative) ring under the usual addition and multiplication of power series, and the leading term of the product of two power series is the product of the leading terms, so we see in particular that $k[[x_1, \ldots, x_r]]$ is an integral domain.

Definition 2.1. Suppose P is a point of an algebraic set X, with $\dim T_P^* := \dim \mathfrak{m}_P/\mathfrak{m}_P^2 = d$, where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{P,X}$. Regular functions $t_1, \ldots, t_d$ are a **system of local parameters** at P if they generate $T_P^*(X)$.

Usually, a system of local parameters is taken at a nonsingular point of X, in which case it behaves closely to the analogous concept in analysis. However, it will be convenient for us to have a term which applies to the more general setting. We then have:

Proposition 2.2. *Let $t_1, \ldots, t_d$ be a system of local parameters at a point P of an algebraic set X. Then sending the x_i to t_i induce a canonical surjective ring homomorphism $k[[x_1, \ldots, x_d]] \to \hat{\mathcal{O}}_{P,X}$.*

Proof. The map sending $\sum_{i=0}^{\infty} F_i(x_1, \ldots, x_d)$ to

$$F_0(t_1, \ldots, t_d), F_0(t_1, \ldots, t_d) + F_1(t_1, \ldots, t_d), F_0(t_1, \ldots, t_d) + F_1(t_1, \ldots, t_d) + F_2(t_1, \ldots, t_d), \ldots$$

is clearly well defined, since $F_n(t_1, \ldots, t_d) \in \mathfrak{m}_P^n$. That it is a homomorphism follows from the definition. For surjectivity, we observe that the hypothesis that the t_i span $\mathfrak{m}_P/\mathfrak{m}_P^2$ implies that monomials of degree n in the t_i span $\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$, which gives the desired statement. □

3. Nonsingular points

The analogy between power series at nonsingular points and Taylor series expansions in analysis is exemplified by the following theorem, which in particular associates to any function regular at P a Taylor series expansion in the t_i:

Theorem 3.1. *Suppose $P \in X$ is a nonsingular point, and $t_1, \ldots, t_d$ a system of local parameters at P. Then the homomorphism $k[[x_1, \ldots, x_d]] \to \hat{\mathcal{O}}_{P,X}$ induced by the t_i is an isomorphism.*

In particular, we obtain an injective homomorphism

$$\mathcal{O}_{P,X} \hookrightarrow k[[x_1, \ldots, x_d]].$$

In order to prove it, we need the following general fact about systems of local parameters, which holds without any nonsingularity hypothesis:

Lemma 3.2. *Let $t_1, \ldots, t_d$ be a system of local parameters for an algebraic set X at a point P. Then the t_i generate $\mathfrak{m}_P$, the maximal ideal of $\mathcal{O}_{P,X}$.*

Proof. Since $\mathcal{O}_{P,X}$ is Noetherian, we know $\mathfrak{m}_P$ is finitely generated, so this is an immediate consequence of Nakayama's lemma. □

Proof of Theorem 3.1. That the last assertion follows from the first is an immediate consequence of Theorem 1.3. For the first assertion, by Proposition 2.2, it is enough to prove injectivity of the homomorphism. This in turn reduces immediately to proving that if $F_n(x_1, \ldots, x_d)$ is homogeneous of degree n, and $F_n(t_1, \ldots, t_d) \in \mathfrak{m}_P^{n+1}$, then we must have $F_n(t_1, \ldots, t_d) = 0$. [This is also equivalent to showing that the monomials of degree n in the t_i are linearly independent in $\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$.]

Accordingly, suppose that we have $F_n(x_1,\dots,x_d)$ homogeneous of degree n with $F_n(t_1,\dots,t_d) \in \mathfrak{m}_P^{n+1}$. Note that if we apply a linear change of variables to the t_i, we obtain a new homogeneous polynomial of degree n which is 0 if and only if the original one was, so we are free to apply any linear change of variables. In particular, if $F_n \neq 0$ we may assume that the x_1^n term of F_n is nonzero. If the coefficient of x_1^n in F_n is $c \in k^*$, then

$$cx_1^n - F_n(x_1,\dots,x_d) \in (x_2,\dots,x_d).$$

On the other hand, it follows by induction from Lemma 3.2 that any element of $\mathfrak{m}_P^{n+1}$ may be written as $\sum_i m_i M_i(t_1,\dots,t_d)$, where $m_i \in \mathfrak{m}_P$ and $M_i(x_1,\dots,x_d)$ is a monomial of degree n. We can thus write

$$F_n(t_1,\dots,t_d) = m_0 t_1^n + \sum_{i>0} m_i M_i(t_1,\dots,t_d),$$

where $m_0 \in \mathfrak{m}_P$ may be 0, and all the M_i are in the ideal $(t_2,\dots,t_d)$. Together with the above, we conclude that $(c-m_0)t_1^d \in (t_2,\dots,t_d)$. But because $m_0 \in \mathfrak{m}_P$ and $c \notin \mathfrak{m}_P$, we have $c - m_0 \notin \mathfrak{m}_P$, and is thus invertible in $\mathcal{O}_{P,X}$. We conclude that in $\mathcal{O}_{P,X}$, we have $t_1^d \in (t_2,\dots,t_d)$. It follows that in a neighborhood of P, we have $Z(t_1) \supseteq Z(t_2,\dots,t_d)$, so that $Z(t_1,\dots,t_d) = Z(t_2,\dots,t_d)$. But by Lemma 3.2, in a neighborhood of P we have $Z(t_1,\dots,t_d) = \{P\}$, which has codimension d in X. On the other hand, by the Krull principal ideal theorem any component of $Z(t_2,\dots,t_d)$ has codimension at most $d-1$ in X, a contradiction. This proves that $F_n = 0$, and hence the desired injectivity. □

4. Completions and singularity type

Theorem 3.1 shows that if P is a nonsingular point of a variety X of dimension d, then $\hat{\mathcal{O}}_{P,X}$ is isomorphic to $k[[x_1,\dots,x_d]]$. This means that the complete local ring plays a role much more analogous to that of a "small neighborhood of P" in topology or classical geometry. Indeed, we have seen that given $P \in X, Q \in Y$ two varieties, then if $\mathcal{O}_{P,X} \cong \mathcal{O}_{Q,Y}$ we have an isomorphism of X and Y on Zariski open subsets, and in particular X and Y are birational. In the topological setting, any two points on two manifolds of the same dimension have homeomorphic open neighborhoods, and the analogous statement we now have in algebraic geometry is that any two nonsingular points on two varieties of the same dimension have isomorphic complete local rings.

It is not hard to see that we also have a converse – if $\hat{\mathcal{O}}_{P,X} \cong k[[x_1,\dots,x_d]]$, then P is nonsingular. This motivates the following definition:

Definition 4.1. Given algebraic sets X, Y and points $P \in X$, $Q \in Y$, P and Q have the same *analytic singularity type* if $\hat{\mathcal{O}}_{P,X} \cong \hat{\mathcal{O}}_{Q,Y}$.

It turns out that this definition agrees extremely well with our intuition about when two singularity "look the same". For instance, we can define:

Definition 4.2. A point P on a curve X is a **node** if it has the same analytic singularity type as the origin in $Z(xy) \subseteq \mathbb{A}^2$.

Exercise 4.3. Show that the origin is a node in the plane curve given by $y^2 = x^3 + x^2$.

5. Applications to components and defining equations

We can now prove two fundamental statements on nonsingular points of algebraic sets.

Theorem 5.1. *Let P be a nonsingular point of an algebraic set X. Then X has a unique irreducible component containing P.*

Proof. It is an immediate consequence of Theorem 3.1 that $\mathcal{O}_{P,X}$ is an integral domain. On the other hand, for any subset of the irreducible components of X, there is a regular function f which vanishes on those components, but no others: this follows from an argument similar to that of Lemma 2.7 of the *Chevalley's theorem and complete varieties* lecture notes. Say Z is an irreducible component of X containing P, and f is not zero on Z, but is zero on all other components of X. Then f is nonzero in $\mathcal{O}_{P,X}$, since any open neighborhood of P contains a nonempty open subset of Z. Thus, if X had two irreducible components containing P, choosing such an f for each of them would give zero divisors in $\mathcal{O}_{P,X}$, a contradiction. □

Remark 5.2. One can also argue directly that if P is a nonsingular point of X, then $\mathcal{O}_{P,X}$ is an integral domain, without the use of power series.

The main consequence of Theorem 5.1, for our purposes, is the following:

Theorem 5.3. *Let P be a nonsingular point of an affine algebraic set $X \subseteq \mathbb{A}^n$ of dimension d, and suppose that $f_1, \dots, f_{n-d} \in I(X)$ are polynomials such that the Jacobian matrix $(\partial f_j / \partial x_i)_{i \in I}$ has full rank $n - d$ at P. Then there exists a Zariski open neighborhood U of P in $\mathbb{A}^n$ such that*

$$Z(f_1, \dots, f_{n-d})|_U = X|_U.$$

Note that by the Jacobian criterion for nonsingularity, such a collection of f_i always exist.

Proof. Since each $f_i \in I(X)$, we have $X \subseteq Z(f_1, \dots, f_{n-d})$. Applying the Jacobian criterion to $Z(f_1, \dots, f_{n-d})$, we find that it is nonsingular at P, of dimension d. Applying Theorem 5.1 to both X and $Z(f_1, \dots, f_{n-d})$, there exists a unique irreducible component of each containing P. Since they have the same dimension, and we have one inclusion, we conclude that the irreducible components are equal, and if U is the complement of any other irreducible components of X and $Z(f_1, \dots, f_{n-d})$, we obtained the desired conclusion. □

Remark 5.4. In fact, our argument for Theorem 5.3 implies a stronger statement: that the f_i actually generate $I(X)$ everywhere locally in U. The result is often stated as "a nonsingular variety is locally a complete intersection."