COMPLEX VARIETIES AND THE ANALYTIC TOPOLOGY

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Classical algebraic geometers studied algebraic varieties over the complex numbers. In this setting, they didn't have to worry about the Zariski topology and its many pathologies, because they already had a better-behaved topology to work with: the analytic topology inherited from the usual topology on the complex numbers themselves. In this note, we introduce the analytic topology, and explore some of its basic properties. We also investigate how it interacts with properties of varieties which we have already defined.

We work throughout in the context of schemes of finite type over a field (generally the complex numbers). However, our considerations will for the most part be topological, so any non-reduced structures will not be relevant to us. If X is of finite type over a field k, we denote by $X(k)$ the set of points of X with residue field k. [It is sometimes useful to recall that this is the same as the set of morphisms $\operatorname{Spec} k \to X$ over $\operatorname{Spec} k$.] Our convention is that a variety over k is an integral separated scheme of finite type over k, and a curve is a variety of dimension 1 (in particular, irreducible). The proofs are largely adapted from [5] and [4].

1. The analytic topology on affine schemes

If $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ is a closed subscheme, we will endow $X(\mathbb{C})$ with a topology which corresponds far more closely than the Zariski topology to our intuition for what X "looks like." We define:

Definition 1.1. The *analytic topology* on X is the topology induced by the inclusion $X(\mathbb{C}) \hookrightarrow \mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^n$, using the usual topology on $\mathbb{C}^n$. The topological space of $X(\mathbb{C})$ endowed with the analytic topology is denoted by X_{an}.

Because zero sets of (multivariate) polynomials are closed in $\mathbb{C}^n$, the analytic topology is finer than the Zariski topology: that is, a closed subset in the Zariski topology is closed in the analytic topology, but not in general vice versa. This may be rephrased into the following conclusion:

Proposition 1.2. *The map of topological spaces $X_{\text{an}} \to X$ induced by the identity on points is continuous.*

Similar arguments also prove the following basic facts.

Exercise 1.3. (a) If also $Y \subseteq \mathbb{A}^m_{\mathbb{C}}$ is a closed subscheme, and $X \to Y$ is a $\mathbb{C}$-morphism, then the induced map $X_{\text{an}} \to Y_{\text{an}}$ is continuous. In particular, a section of $\mathcal{O}_X(X)$ induces a continuous map $X_{\text{an}} \to \mathbb{C} = (\mathbb{A}^1_{\mathbb{C}})_{\text{an}}$.

(b) The analytic topology on X is an isomorphism invariant, independent of the particular imbedding of X into affine space.

(c) If $Z \subseteq X$ is a subscheme, then $Z_{\text{an}} = (X_{\text{an}})|_{Z(\mathbb{C})}$.

We will need one elementary result on the continuity of roots of a single-variable complex polynomial.

Theorem 1.4. *Let $f(x) = a_0 + a_1 x + \cdots + a_d x^d$ be a nonzero complex polynomial, and let $c \in \mathbb{C}$ be a root of f. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $b_0, \ldots, b_d \in \mathbb{C}$ with $|a_i - b_i| < \delta$ for all i, there is a root c' of the polynomial $g(x) = b_0 + b_1 x + \cdots + b_d x^d$ with $|c - c'| < \epsilon$.*

Proof. Let γ be a circle of radius less than ϵ around c, chosen so that there are no other zeros of $f(x)$ on γ. Let w be the minimum absolute value of $f(x)$ on γ, which is strictly positive by construction. For δ sufficiently small, we have that if $b_0, \ldots, b_d \in \mathbb{C}$ satisfy $|a_i - b_i| < \delta$, then $|f(x) - g(x)| < w$ for all $x \in \gamma$, where $g(x) = b_0 + b_1 x + \cdots + b_d x^d$. It follows from Rouche's theorem that f and g have the same number of roots inside the circle γ, which gives the desired statement. $\square$

We can now begin to make statements about the analytic topology of varieties, at least in some special cases.

Corollary 1.5. *Let $C \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a curve in the plane. Then C_{an} has no isolated points.*

Proof. Since C has codimension 1 in $\mathbb{A}^2_{\mathbb{C}}$, we know it can be expressed as the zero set of a single polynomial, say $f(x,y) \in \mathbb{C}[x,y]$, with $\deg f = d$. Then $f(x,y)$ has degree at most d when considered as a polynomial in y, and its coefficients are themselves continuous functions of x. We may assume that $f(x,y)$ is not constant in y, since otherwise by irreducibility of C we must have $f = x - c$ for some c, so C is a vertical line and certainly has no isolated points. Thus, given $(x_0, y_0) \in C$, it follows from Theorem 1.4 that for any $\epsilon > 0$, there is some $\delta > 0$ such that for every x with $|x - x_0| < \delta$, there is some y with $|y - y_0| < \epsilon$ and $f(x,y) = 0$. We conclude that (x_0, y_0) is not an isolated point of C. $\square$

Example 1.6. To see that the above Corollary has some content to it, we observe that it is false over the real numbers. Indeed, the curve $y^2 = x^3 - x^2$ is irreducible over $\mathbb{R}$, but its real points consist of a single one-dimensional component supported on $x \geqslant 1$, and a single isolated point at the origin.

2. The analytic topology on schemes

Having defined the analytic topology on affine schemes, we can now define it on arbitrary schemes, via gluing.

Definition 2.1. Let X be a scheme of finite type over $\mathbb{C}$. The *analytic topology* on X, denoted by X_{an}, is the topology such that for each affine open subset U, the natural map $U_{\text{an}} \to (X_{\text{an}})|_{U(\mathbb{C})}$ is a homeomorphism.

Once again, there are some basic properties to check for the analytic topology:

Exercise 2.2. (a) The analytic topology X_{an} on a prevariety X is well defined.

(b) The map of topological spaces $X_{\text{an}} \to X$ induced by the identity on points is continuous.

(c) Given another scheme Y of finite type over $\mathbb{C}$, and a $\mathbb{C}$-morphism $\varphi : X \to Y$, the induced map $X_{\text{an}} \to Y_{\text{an}}$ is continuous.

(d) If $Z \subseteq X$ is a subscheme, then $Z_{\text{an}} = (X_{\text{an}})|_{Z(\mathbb{C})}$.

Example 2.3. Complex projective space $\mathbb{P}^n_{\mathbb{C}}$ (which is just the classical $\mathbb{CP}^n$) is compact in the analytic topology. Indeed, we have the natural morphism $\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$, which by Exercise 2.2 (c) induces a continuous map $(\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\})_{\text{an}} \to (\mathbb{P}^n_{\mathbb{C}})_{\text{an}}$. Inside $(\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\})_{\text{an}}$ we have the closed subset consisting of elements of norm 1; identifying $\mathbb{A}^{n+1}_{\mathbb{C}}$ with $\mathbb{R}^{2n+2}$, this subset is precisely the unit sphere, and hence compact. Moreover, it surjects onto $(\mathbb{P}^n_{\mathbb{C}})_{\text{an}}$, and since the continuous image of a compact set is compact, we conclude that $(\mathbb{P}^n_{\mathbb{C}})_{\text{an}}$ is compact.

As an immediate consequence of Example 2.3 and Exercise 2.2, we conclude:

Corollary 2.4. *If X is a projective scheme over $\mathbb{C}$, then X_{an} is compact.*

We know that if X, Y are schemes over $\mathbb{C}$, the Zariski topology on $X \times_{\operatorname{Spec} \mathbb{C}} Y$ is not the product topology. However, (again confirming that the analytic topology behaves closer to our intuition) for the analytic topology we have:

Exercise 2.5. If X, Y are schemes of finite type over $\mathbb{C}$, then

$$(X \times_{\operatorname{Spec}\mathbb{C}} Y)_{\mathrm{an}} = X_{\mathrm{an}}) \times (Y_{\mathrm{an}},$$

where on the righthand side we take the product topology.

Note that by the universal property of fibered products, we at least have $(X \times_{\operatorname{Spec}\mathbb{C}} Y)(\mathbb{C})$ canonically identified with $X(\mathbb{C}) \times Y(\mathbb{C})$.

We can now conclude that separatedness implies that the analytic topology is Hausdorff.

Corollary 2.6. *If X is a separated scheme of finite type over $\mathbb{C}$, then X_{an} is Hausdorff.*

Proof. For X_{an} to be Hausdorff is equivalent to the image $\Delta(X_{\mathrm{an}})$ in $X_{\mathrm{an}} \times X_{\mathrm{an}}$ to be closed. Since X is separated, $\Delta(X)$ is closed in $X \times X$ in the Zariski topology, so $\Delta(X_{\mathrm{an}})$ is closed in $(X \times X)_{\mathrm{an}}$ by Exercise 2.2 (b). But $(X \times X)_{\mathrm{an}} = X_{\mathrm{an}} \times X_{\mathrm{an}}$ by Exercise 2.5, so we conclude that X_{an} is Hausdorff. □

In fact, we will see in Corollary 3.3 that the converse also holds: if X is a complex prevariety and X_{an} is Hausdorff, then X is a variety. However, this requires putting together some of the deeper results which we have developed.

We can generalize Corollary 1.5 as follows:

Corollary 2.7. *Let X be a complex curve, or more generally an irreducible one-dimensional scheme of finite type over $\mathbb{C}$. Then X_{an} has no isolated points.*

Before we give the proof, we note that the analytic topology on a prevariety X is locally metrizable (and indeed metrizable if X is separated, from Corollary 2.6). Thus we can work more conceptually with sequential limits and compactness.

Proof. We prove the statement in several steps. We first prove it for projective nonsingular curves $X \subseteq \mathbb{P}^n_{\mathbb{C}}$. Given $P \in X$, we know that there exists a morphism $\varphi : X \to \mathbb{P}^2_{\mathbb{C}}$ such that $\varphi^{-1}(\varphi(P)) = \{P\}$. Since $\varphi(X)$ is a (projective) plane curve, noting that we can replace it by an affine open neighborhood of $\varphi(P)$, we know by Corollary 1.5 that $\varphi(P)$ is not an isolated point of $\varphi(X)_{\mathrm{an}}$. Now, let $Q_1, Q_2, \dots$ be a sequence of points of $\varphi(X)$ approaching $\varphi(P)$ in the analytic topology, and lift them to $P_1, P_2, \dots$ in X. We know that X_{an} is compact by Corollary 2.4, so some subsequence of $P_1, P_2, \dots$ converges to a point $P' \in X$ in the analytic topology. But then since φ is continuous, we conclude $\varphi(P') = \varphi(P)$, so $P = P'$, and P is not an isolated point of X.

The case of an arbitrary nonsingular curve X then follows, since we know that X can be realized as a Zariski open subset of some nonsingular projective $\bar{X}$, and $\bar{X} \smallsetminus X$ consists of finitely many points, so if $\bar{X}_{\mathrm{an}}$ has no isolated points then X_{an} cannot have any isolated points either.

Next, we conclude the statement of the corollary for an arbitrary curve X by considering the normalization morphism $\nu : \tilde{X} \to X$; this is a surjective morphism, with $\tilde{X}$ a nonsingular curve. For any $P \in X$, we have $\nu^{-1}(P)$ a closed set, hence a finite set of points, and since we know that $\tilde{X}_{\mathrm{an}}$ has no isolated points, if we choose a sequence in $\tilde{X}$ converging to a point of $\nu^{-1}(P)$, it contains at most finitely many points in $\nu^{-1}(P)$, and its image is then a sequence converging to P, so X_{an} does not have any isolated points.

Finally, reducedness is irrelevant since the question is topological, and separatedness is likewise not an issue, as an isolated point would remain isolated in an affine (hence separated) open neighborhood, so we conclude the more general statement as well. □

3. Separatedness and properness

There are a number of basic and important facts relating the ideas we have introduced for schemes to standard topological properties applied to the analytic topology. The main ingredient for proving these statements is the following:

Theorem 3.1. *Let X be a scheme of finite type over $\mathbb{C}$, and U a Zariski-dense open subset. Then U_{an} is dense in X_{an}.*

Proof. We first observe that the statement of the theorem in the case that X is a curve is precisely Corollary 2.7. Now, for arbitrary X, given $P \in X \smallsetminus U$, by Lemma 3.1 of *Properties of fibers and applications* there exists a one-dimensional integral closed subscheme Z of X with $P \in Z$ and $Z \cap U \neq \emptyset$. Restricting Z to an open neighborhood of P, we may assume Z is separated, hence a curve. By Exercise 2.2 (d), we have $Z_{\mathrm{an}} = (X_{\mathrm{an}})|_{Z(\mathbb{C})}$, and P is in the (analytic) closure of $Z_{\mathrm{an}} \cap U$ since we already that the result hold for curves. It thus follows that P is in the closure of U in X_{an}, and we conclude that U is dense, as asserted. □

In particular, we conclude the following:

Corollary 3.2. *Let X be a scheme of finite type over $\mathbb{C}$, and $Z \subseteq X$ a subscheme. If Z_{an} is closed in X_{an}, then Z is a closed subscheme of X.*

Proof. It is enough to show that Z is closed in X in the Zariski topology. The question being topological, we may assume that Z is reduced, and then if $\bar{Z}$ denotes the closure in X, we have from the definition of subscheme that Z is Zariski open (and dense) in $\bar{Z}$, so by Theorem 3.1, we conclude that Z_{an} is dense in $\bar{Z}_{\mathrm{an}}$. Then if Z_{an} is closed in X_{an}, we conclude that $Z_{\mathrm{an}} = \bar{Z}_{\mathrm{an}}$, and hence $Z(\mathbb{C}) = \bar{Z}(\mathbb{C})$. Now, $\bar(Z) \smallsetminus Z$ is of finite type over $\mathbb{C}$, so if nonempty, it would have to contain a point of $\bar{Z}(\mathbb{C})$. We thus conclude that $Z = \bar{Z}$, as desired. □

We can now prove several basic statements on the analytic topology.

Corollary 3.3. *Let X be of finite type over $\mathbb{C}$. Then X is separated over $\mathbb{C}$ if and only if X_{an} is Hausdorff.*

Proof. One direction was proved already in Corollary 2.6. Conversely, suppose X_{an} is Hausdorff, so that the diagonal $\Delta(X)$ is closed in the analytic topology. Then by Corollary 3.2, we have $\Delta(X)$ closed also in the Zariski topology, so X is separated. □

To apply Theorem 3.1 to study compactness, we will want a more general form of Chow's lemma than is stated in [2].

Lemma 3.4. *Let S be a Noetherian scheme, and X separated and of finite type over S. Then there exists a scheme X' and a surjective projective morphism $f : X' \to X$ such that X' is quasi-projective over S, and there is a dense open subset $U \subseteq X$ such that $f|_U : f^{-1}(U) \to U$ is an isomorphism.*

The proof is essentially the same; see Theorem 5.6.1 of [1]. Note that the form in [2] follows from this, since if X is proper over S, then X' is both quasi-projective and proper over S, and it follows that X' is projective over S.

Corollary 3.5. *A separated scheme X of finite type over $\mathbb{C}$ is proper over $\mathbb{C}$ if and only if X_{an} is compact.*

Proof. First suppose that X_{an} is compact, and X is quasiprojective. Then given an immersion $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, we have that the image of X_{an} is closed in $(\mathbb{P}^n_{\mathbb{C}})_{\mathrm{an}}$, so by Corollary 3.2 we have that X is a closed subscheme, hence projective. In the general case, let $f : X' \to X$ be as in Chow's lemma. Since f is projective and X_{an} is compact, we conclude that $(X')_{\mathrm{an}}$ is compact, hence projective over $\mathbb{C}$. Finally, because surjectivity is closed under base change, we see easily that since X' is universally closed over $\mathbb{C}$, we must have X universally closed as well, hence proper.

For the converse, first suppose that X is projective. Then X_{an} is compact by Corollary 2.4. Now suppose instead that X is proper. Then again using Lemma 3.4, there is a surjective morphism $X' \to X$ for some projective variety X'. Thus, X_{an} is the continuous image of the compact space $(X')_{\mathrm{an}}$, and is therefore compact. □

4. A digression on nonsingularity

In order to conclude our study of the complex topology, we will need to know an important and basic fact: a nonsingular complex curve has the natural structure of a (one-dimensional) complex manifold. We will prove the more general statement for complex varieties of any dimension. Before doing so, we need to develop a few basic ideas that apply over any field.

Definition 4.1. Let X be a scheme of finite type over a field k, and $x \in X$ a nonsingular point, at which $\dim X = d$. A **system of local coordinates** for X and x is a d-tuple $(t_1, \dots, t_d)$ of elements of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ which gives a basis for the Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Note that by Nakayama's lemma, $(t_1, \dots, t_d)$ is a system of local coordinates if and only if the t_i generate the maximal ideal. We will need the Jacobian criterion for nonsingularity, which we state in a slightly special case below:

Theorem 4.2. *Let $X \subseteq \mathbb{A}_k^n$ be an affine scheme given by an ideal $I \subseteq k[t_1, \dots, t_n]$. Given $x \in X(k)$, suppose that $\dim \mathcal{O}_{X,x} \geqslant d$. Let $c_1, \dots, c_n \in k$ be the values of the t_i at x. Then the following are equivalent:*

1. *X is nonsingular at x of dimension d, and $(t_{n-d+1} - c_{n-d+1}, \dots, t_n - c_n)$ induces a system of local coordinates for X at x;*
2. *there exist $f_1, \dots, f_{n-d} \in I$ such that the Jacobian matrix $\left(\frac{\partial f_i}{\partial t_j}\right)_{1 \leqslant i,j \leqslant n-d}$ is invertible at x.*

Furthermore, in this case we have that $f_1, \dots, f_{n-d}$ generates I in a neighborhood of x.

Proof. Denote by $T_x^* X$ and $T_x^* \mathbb{A}_k^n$ the Zariski cotangent spaces at x of X and $\mathbb{A}_k^n$, respectively (that is, the maximal ideal modulo its square in the local rings at x). Given $f \in k[t_1, \dots, t_n]$, denote by df_x the element of $T_x^* \mathbb{A}_k^n$ induced by $f - f(c_1, \dots, c_n)$. Since $\kappa(x) = k$, we have that both $T_x^* X$ and $T_x^* \mathbb{A}_k^n$ spaces are k-vector spaces, with the latter of dimension n and having basis given by the $d(t_i)_x$. We also have the natural surjection $\pi : T_x^* \mathbb{A}_k^n \to T_x^* X$, whose kernel we observe is precisely the image of I in $T^* \mathbb{A}_k^n$. Furthermore, we know that $\dim T_x^* X \geqslant d$, with equality if and only if X is nonsingular of dimension d at x. Thus, we obtain the following equivalences:

- (1) holds;
- the images under π of $d(t_{n-d+1})_x, \dots, d(t_n)_x$ span the space $T_x^* X$;
- the kernel of π has dimension $n - d$, and is disjoint from the span of $d(t_{n-d+1})_x, \dots, d(t_n)_x$;
- the kernel of π maps isomorphically onto the span of $d(t_1)_x, \dots, d(t_{n-d})_x$ under projection to the first $n - d$ coordinates;
- the kernel of π surjects onto the span of $d(t_1)_x, \dots, d(t_{n-d})_x$ under projection to the first $n - d$ coordinates;
- there exist $f_1, \dots, f_{n-d} \in I$ such that the span of the $(df_i)_x$ surjects onto the span of $d(t_1)_x, \dots, d(t_{n-d})_x$ under projection to the first $n - d$ coordinates.

We then observe that given any $f_1, \dots, f_{n_d}$ vanishing at x, the map $k^{n-d} \to k^{n-d}$ given by projecting each $(df_i)_x$ to the span of $d(t_1)_x, \dots, d(t_{n-d})_x$ is given by the Jacobian matrix of condition (2). Thus, we conclude that the final condition above is equivalent to (2), as desired.

For the last assertion, we cheat a bit, insofar as we will invoke the fact (which we haven't proved) that regular local rings are integral domains. Let $J = (f_1, \dots, f_{n-d})$, and set $Y \subseteq \mathbb{A}_k^n$ to be the closed subscheme on which J vanishes, so that X is a closed subscheme of Y. Then by the Krull principal ideal theorem, Y has dimension at least d at all its closed points, and in particular at x. Applying the equivalence of (1) and (2), we conclude that Y is nonsingular of dimension d at x. Now we use that both $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,x}$ are integral domains: the generic point of X must correspond to a prime ideal of $\mathcal{O}_{Y,x}$, but in a neighborhood of x, but we have that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,x}$, so the generic point of X must correspond to the zero ideal in $\mathcal{O}_{Y,x}$, so $\mathcal{O}_{X,x} = \mathcal{O}_{Y,x}$. We thus have that $f_1, \dots, f_{n-d}$ generate I in $\mathcal{O}_{\mathbb{A}^n_k,x}$. We finally conclude (using that I is finitely generated) that the f_i generate I in a neighborhood of x. $\square$

Remark 4.3. In addition to standard commutative algebra citations for regular local rings being integral domains (for instance, Theorem 14.3 of [3]), one can prove the result in the case we need by developing the theory of completions and power series; see Theorem 3.1 of the notes *Power series and nonsingular points* from Math 248B, Winter 2010.

Just for fun, we mention the following corollary, which says that nonsingular points are "locally complete intersections":

Corollary 4.4. *Let X be of finite type over k, and $x \in X(k)$ a nonsingular point of X of dimension d. Then for any neighborhood U of x, and any closed immersion $U \subseteq \mathbb{A}^n_k$ with U defined by an ideal I, there exists $f_1, \dots, f_{n-d} \in I$ such that the f_i generate I in an open neighborhood of x.*

Proof. If $\mathbb{A}^n_k$ has coordinates $t_1, \dots, t_n$, and in these coordinates $x = (c_1, \dots, c_n)$, then since the natural map $\mathcal{O}_{\mathbb{A}^n_k,x} \to \mathcal{O}_{X,x}$ is a surjection, we have that the $t_i - c_i$ generate the maximal ideal of $\mathcal{O}_{X,x}$, and some d of them must give a system of local coordinates for X at x. Without loss of generality, we may assume these are $t_{n-d+1} - c_{n-d+1}, \dots, t_n - c_n$. We then conclude the desired statement from Theorem 4.2. $\square$

We now return to the complex case with the following corollary of Theorem 4.2:

Corollary 4.5. *Let X be a scheme of finite type over $\mathbb{C}$, and $x \in X(\mathbb{C})$ a nonsingular point of X. If $(t_1, \dots, t_d)$ is a system of local coordinates for X at x, there exists a neighborhood U of x in X_{an} on which all the t_i induce sections of $\mathcal{O}_X$, and such that the induced map $U \to \mathbb{C}^d$ defines a homeomorphism of U_{an} onto an open neighborhood V of the origin in $\mathbb{C}^d$, with every element of $\mathcal{O}_X$ corresponding to an analytic function on the image of its domain of definition in V.*

In particular, if X is a nonsingular variety of dimension d, then X_{an} has the structure of a complex manifold of dimension d, with sections of $\mathcal{O}_X$ giving rise to analytic functions on X_{an}.

Proof. The first statement being local on X, we may assume that $X = \operatorname{Spec} A$ is affine and the t_i are global sections of $\mathcal{O}_X$. Since X is of finite type, for some n we can extend the t_i to a set of n generators of A over $\mathbb{C}$, and let I be the corresponding ideal of definition for X in $\mathbb{A}^n_{\mathbb{C}}$. Reindexing and applying Theorem 4.2, we conclude that I is generated in a neighborhood of x by some $f_1, \dots, f_{n-d}$, and that $\left(\frac{\partial f_i}{\partial t_j}\right)_{1 \leqslant i,j \leqslant n-d}$ is invertible at x. By the implicit function theorem, there exist neighborhoods U of x in X_{an}, V of the origin in $\mathbb{C}^d$ and an analytic function $g : V \to \mathbb{C}^{n-d}$ such that $\text{id} \times g : V \to \mathbb{C}^n$ maps onto U and is inverse to the projection map $\mathbb{C}^n \to \mathbb{C}^d$. This gives the desired homeomorphism. In addition, g expresses all the t_i as analytic functions of $t_1, \dots, t_d$, and since sections of $\mathcal{O}_X$ are expressed locally as rational functions in the t_i with denominators nonvanishing on their domain of definition, we conclude that they are analytic on their domain of definitions.

For the final statement, if X is a variety it is in particular separated, so X_{an} is Hausdorff by Corollary 2.6. The first statement gives us a complex atlas on an open cover of X_{an}, and to conclude that the induced transition maps are analytic, we need only use that algebraic isomorphisms are analytic under above correspondence, which follows easily from the fact that sections of $\mathcal{O}_X$ yield analytic functions. $\square$

5. A digression on the Riemann-Roch theorem

We will need to know the following fact:

Theorem 5.1. *Let X be a nonsingular projective curve over a field k, and $x \in X$ a closed point. Then there exists a (nonconstant) rational function $f \in K(X)$ with a pole only at x.*

This is an immediate consequence of the Riemann-Roch theorem; see for instance Theorem IV.1.3 of [2]. However, if one only wants to prove this type of statement, it suffices to prove an inequality, which is easier. See §III.6.6, Lemma of [5].

Remark 5.2. A stronger but closely related version of Theorem 5.1 is that if X is a nonsingular projective curve over k, and $x \in X$ a closed point, then there exists a closed immersion of X into $\mathbb{P}^n_k$ for some n, such that there exists a hyperplane H which intersects the image of X only in the image of x.

One can show that an open subset of an affine nonsingular curve is still affine, and it follows from this and the above that any nonsingular curve which is not projective is in fact affine.

6. Connectedness

We conclude our discussion of the complex topology with the following foundational theorem:

Theorem 6.1. *Let X be of finite type over $\mathbb{C}$. Then X is connected in the Zariski topology if and only if X_{an} is connected.*

As we shall see, this theorem is quite a bit deeper than the corresponding statements for properness and separatedness. This difficulty should perhaps not be surprising: if we consider the analogous statement over $\mathbb{R}$, it remains true that a separated scheme is Hausdorff in the real analytic topology, and a proper scheme is compact, but it is not true that a (Zariski) connected scheme is connected in the real analytic topology, as we see already with elliptic curves in the plane. Nonetheless, with the tools we have developed the proof of the theorem will not be very difficult.

Proof. Since the analytic topology is finer than the Zariski topology, it is clear that if X_{an} is connected, then so is X.

For the converse, we begin by proving that the desired result holds in the case that X is a nonsingular projective curve. Given $x \in X$, we know from Theorem 5.1 that there exists a (nonconstant) rational function f with a pole only at x. We know by Corollary 3.5 and Corollary 4.5 that X_{an} is a compact one-dimensional complex manifold, and f (being a quotient of analytic functions) induces a meromorphic function on X_{an}. If X_{an} is disconnected, let $C \subseteq X_{\mathrm{an}}$ be a connected component not containing x. Then f is analytic on C, so the maximum modulus principle implies that f is constant on C. But then subtracting this constant we obtain a non-zero rational function with infinitely many zeroes on X, which is impossible. Thus, X_{an} is connected.

Next, if X is an arbitrary nonsingular curve, we know that it may be imbedded as an open subset of a nonsingular projective curve $\overline{X}$. We now know that $\overline{X}_{\mathrm{an}}$ is a connected complex manifold of dimension 1. But the complement of X_{an} in $\overline{X}_{\mathrm{an}}$ is a finite set of points, so we conclude that X_{an} is likewise connected, as desired.

Now suppose that X is any curve, and let $\tilde{X} \to X$ be the normalization. Then we have shown that $\tilde{X}_{\mathrm{an}}$ connected, but $\tilde{X}_{\mathrm{an}}$ surjects onto X_{an}, so we conclude that X_{an} is likewise connected. Finally, suppose that X is arbitrary of dimension 1. Inducting on the number of components, we reduce to the case that X is irreducible. Passing to an affine open subset, we omit only finitely many points, which by Corollary 2.7 cannot be isolated, so connnectedness is not affected. We may thus assume that X is separated, and since reducedness is irrelevant, we have reduced to the previously addressed case that X is a curve.

To address the case of arbitrary dimension, we have proved in Theorem 3.2 of *Properties of fibers and applications* that any two points of $X(\mathbb{C})$ lie in some connected 1-dimensional closed subscheme Z. Since we now know that Z_{an} is connected, we conclude that X_{an} is connected, as desired. □

References

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