PROPERTIES OF FIBERS AND APPLICATIONS

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We discuss properties of schemes over a field, and of fibers of morphisms of finite type. We apply these results to prove a theorem which can be thought of as saying that every connected abstract algebraic set over a field is path connected.

1. Schemes over a field

Although schemes over a field are in some sense a very special case, they are of central importance for two reasons. First, obviously, is that classical algebraic geometry is focused on them. Second is that if $f : X \to Y$ is any morphism, no matter how pathological X and Y may be, the fibers of f are, by definition, schemes over a field.

Our main motivation is to study their basic properties, and in particular how they behave under extension of the base field. We quickly see that many of the most basic properties are not preserved under base extension. To save ink and/or electrons, we will frequently write k instead of $\operatorname{Spec} k$ when k is a field.

Example 1.1. Let $k = \mathbb{C}(t)$, and set $X = \operatorname{Spec} k[x]/(x^2 - t)$. The polynomial $x^2 - t$ is irreducible over k, so X is the spectrum of a field, and in particular is irreducible. But if we let $k' = \mathbb{C}(\sqrt{t})$, then $X' = X \times_k k'$ is $\operatorname{Spec} k'[x]/(x^2 - t)$, and $x^2 - t$ now factors over k', so in fact $k'[x]/(x^2 - t) \cong k' \times k'$, and X' is the disjoint union of two points. Thus, we see that irreducibility and connectedness are not preserved under base extension for scheme of finite type over a field.

Example 1.2. Let $k = \mathbb{F}_p(t)$, and set $X = \operatorname{Spec} k[x]/(x^p - t)$. As before, $x^p - t$ is irreducible, so X is a single reduced point. But if $k' = \mathbb{F}_p(\sqrt[p]{t})$, and $X' = X \times_k k'$, then $X' = \operatorname{Spec} k'[x]/(x^p - t) = \operatorname{Spec} k'[x]/(x - \sqrt[p]{t})^p$, so we see that X' is non-reduced.

The main theorem regarding this sort of behavior is the following, which says in essense that the above examples are the only things that can go wrong under extension of base fields. Before giving the statement, we recall some terminology. We assume familiarity with separable and inseparable field extensions in the algebraic case.

Definition 1.3. A field k is **algebraically separably closed** if it has no nontrivial separable algebraic extensions. It is **algebraically separably closed** in an extension field k' if every element of k' which is algebraic and separable over k is in fact contained in k. A field k is **perfect** if it has no inseparable algebraic extensions.

Finally, a (not necessarily algebraic) field extension k'/k is **separable** if either $\operatorname{char} k = 0$ or $\operatorname{char} k = p$ and k' is linearly disjoint from $k^{1/p}$ inside $\overline{k'}$, where $k^{1/p}$ denotes the subfield of $\overline{k'}$ generated by pth roots of elements of k. Recall that linearly disjoint means that the tensor product $k' \otimes_k k^{1/p}$ injects into the compositum $(k')(k^{1/p}) \subseteq \overline{k'}$.

In particular, if $\operatorname{char} k = 0$, then algebraically separably closed is equivalent to algebraically closed, while every field is perfect.

In general, if k' has a transcendence basis $\{t_i\}_{i \in I}$ over k such that k' is separable over $k(\{t_i\}_{i \in I}$, then k' is separable over k (the converse is also true if K is finitely generated over k); see §26 of [Mat86].

Theorem 1.4. *Let X be a scheme over a field k. Then the following are equivalent:*

1. *$X \times_k \bar{k}$ is irreducible;*
2. *$X \times_k k'$ is irreducible, for some separably algebraically closed field extension k' of k;*
3. *$X \times_k k'$ is irreducible for every field k' extending k;*
4. *X is irreducible, and if k' is the residue field at the generic point of X, we have that k is separably algebraically closed in k'.*

In addition, the following are equivalent:

1. *$X \times_k \bar{k}$ is reduced;*
2. *$X \times_k k'$ is reduced, for some perfect field extension k' of k;*
3. *$X \times_k K$ is reduced for all extension fields K of k;*
4. *X is reduced, and for every k' which is the residue field at a generic point of X, we have that k' is separable over k.*

The proof is somewhat involved. See Exercise II.3.15 of [Har77], as well as Propositions 4.5.1, 4.5.9, 4.5.21 and 4.6.1 of [GD65]. Intuitively, the point is that phenomena relating to topology can only change under separable extensions, while phenomena relating to algebra can only change under inseparable extensions.

Thus motivated, we define:

Definition 1.5. A scheme X over k is **geometrically irreducible** (respectively **geometrically reduced**; **geometrically integral**) if it satisfies the first (respectively second; first and second) collection of equivalent properties in Theorem 1.4.

The terminology "geometric" is frequently used to describe objects or properties which involve an algebraic closed field.

Remark 1.6. There is also a version of Theorem 1.4 for connectedness. However, there is no equivalent statement for (4), because unlike irreducibility or reducedness, connectedness need not be preserved under restriction to an open subset.

2. Fibers of morphisms of finite type

Part of the promise of schemes was that generic points should be relevant to open subsets in nice situations. One natural situation that arises is in studying fibers of morphisms: it is natural to wonder if, for instance, the generic fiber being irreducible or reduced implies the same for fibers on some nonempty open subset. We quickly see that this is not the case.

Example 2.1. Let k be an algebraically closed field, and consider $X = \operatorname{Spec} k[x], Y = \operatorname{Spec} k[y]$, and the morphism $f : X \to Y$ given by $f^\sharp(y) = x^2$. Then the generic fiber is simply the point $\operatorname{Spec} k(x)$ over $\operatorname{Spec} k(y)$, with $k(x)$ considered as a degree-2 extension of $k(y)$ by setting $y = x^2$. In particular, the generic fiber is integral. On the other hand, none of the closed fibers are integral. If $\operatorname{char} k \neq 2$, then the fibers over $(y-a)$ for $a \neq 0$ consist of two points each, and are not irreducible, while the fiber over (y) is is a single reduced point. If $\operatorname{char} k = 2$, on the other hand, all of the fibers are single non-reduced points.

On the other hand, we observe that the problem in these examples is that while the generic fiber is integral, it is not geometrically integral. We thus suspect that to achieve good behavior of properties in fibers, we should consider their "geometric" versions. This suspicion is borne out by the following fundamental theorem:

Theorem 2.2. *Suppose $f : X \to Y$ is a morphism of finite type, with Y irreducible. Suppose the generic fiber of f is geometrically irreducible (respectively, geometrically reduced, geometrically integral). Then there exists a nonempty open subset $U \subseteq Y$ such that the fiber X_y is geometrically irreducible (respectively, geometrically reduced, geometrically integral) for all $y \in U$.*

See Theorem 9.7.7 of [GD66] for the case f is of finite presentation. The general case may be reduced to this case via a strong form of generic freeness; see the Stacks Project, tag 052B. We do not describe the proof in detail, but we do mention that ultimately, it reduces to the following classical fact, which is essentially the case of an affine hyperplane. We do explain the proof of the proposition, because the idea behind it is simple and clear, and it uses several techniques of frequent importance in algebraic geometry.

Proposition 2.3. *Let A be a ring, and $f \in A[x_1, \ldots, x_n]$ a polynomial of degree d. Then there is a closed subset $Z \subseteq \operatorname{Spec} A$ such that for any prime ideal $\mathfrak{p}$ of A the following are equivalent:*

(1) $\mathfrak{p} \in Z$;
(2) *there exists an algebraically closed field extension k of the residue field $\kappa(\mathfrak{p})$ such that either f is reducible or has degree strictly less than d when considered in the natural way as a polynomial over k;*
(3) *for every algebraically closed field extension k of $\kappa(\mathfrak{p})$, either f is reducible or has degree strictly less than d when considered as a polynomial over k.*

Note that if f is the zero polynomial over k, we consider it to have degree strictly less than d. Also note that the statement would be false with only the reducibility, as for instance demonstrated by the polynomial $tx^2 + x$ in the case $A = k[t]$ for some field k.

Remark 2.4. Contained in Proposition 2.3 is the case that A is a field, which is none other than the particular case of Theorem 1.4 that for affine hypersurfaces, integrality over all algebraically closed field extensions is equivalent to integrality for any algebraically closed field extension.

The following lemma will be used in our proof of the proposition, and illuminates why it is more natural to consider properties of geometric fibers rather than of the fibers themselves:

Lemma 2.5. *Let $f : X \to Y$ be a morphism of schemes, locally of finite type. Given $y \in Y$, the following are equivalent:*

(1) $y \in f(X)$;
(2) *there exists an algebraically closed extension field k of $\kappa(y)$ such that the associated morphism $\operatorname{Spec} k \to Y$ with image y factors through f;*
(3) *for any algebraically closed extension field k of $\kappa(y)$, the associated morphism $\operatorname{Spec} k \to Y$ with image y factors through f.*

Recall (Exercise II.2.7 of [Har77]) that for any morphism Y and any field k, a morphism $\operatorname{Spec} k \to Y$ is equivalent to a point $y \in Y$ together with a field injection $\kappa(y) \hookrightarrow k$.

Proof. It is clear that (3) implies (2) which implies (1). Conversely, if $y \in f(X)$, then the fiber $X_y := X \times_Y \operatorname{Spec} \kappa(y)$ is nonempty. Now, X_y is locally of finite type over $\operatorname{Spec} \kappa(y)$, so it contains a nonempty open subset which is of finite type, and in particular contains some closed point x. Then $\kappa(x)$ is a finite extension of $\kappa(y)$, so if k is any algebraically closed extension of $\kappa(y)$, we may choose an inclusion $\kappa(x) \hookrightarrow k$ compatible with the previously chosen inclusions of $\kappa(y)$ into each, and we obtain a morphism of the desired form by composing with the canonical morphism $\operatorname{Spec} \kappa(x) \to X$ with image x. We thus conclude (3). $\square$

The next lemma is a nice application of our work on properness, and will be important as well:

Lemma 2.6. *Given a ring A, and integers $n, m, n_1, \ldots, n_m$, suppose a morphism $f : \prod_i \mathbb{A}_A^{n_i} \to \mathbb{A}_A^n$ over $\operatorname{Spec} A$ is given by n polynomials, such that:*

- *there exist $d_1, \ldots, d_m$ such that each polynomial is homogeneous of degree d_i in the variables corresponding to the ith multiplicand;*

- *we have*

$$f^{-1}(0) \subseteq \bigcup_{i=1}^{m} \mathbb{A}_A^{n_1} \times \cdots \times \mathbb{A}_A^{n_{i-1}} \times (0) \times \mathbb{A}_A^{n_{i+1}} \times \cdots \times \mathbb{A}_A^{n_m}.$$

Then the image of f is closed in $\mathbb{A}_A^n$.

Proof. The hypotheses imply that f induces a morphism $\bar{f} : \prod_i \mathbb{P}_A^{n_i-1} \to \mathbb{P}_A^{n-1}$ over $\operatorname{Spec} A$, which has closed image because the source is proper and the target is separated over $\operatorname{Spec} A$. Denote by Z_n the zero section of $\mathbb{A}_A^n$ (that is, the closed subscheme represented by $\operatorname{Spec} A \to \mathbb{A}_A^n$ obtained by sending all variables to 0). Again using homogeneity, the image of f contains Z_n. But using the canonical morphisms $\pi_r : \mathbb{A}_A^r \smallsetminus Z_r \to \mathbb{P}_A^r$, we see that the image of f on the complement of Z_n is precisely the preimage under π_n of the image of $\bar{f}$. Thus, the image of f is closed after restriction to the complement of Z_n, and it contains Z_n, so it must be closed. ☐

Proof of Proposition 2.3. We first observe that the desired statement is "functorial", in the following sense: let $\varphi : A' \to A$ be a ring homomorphism, and suppose that f is obtained from f' by applying φ to the coefficients. Further suppose we have $Z' \subseteq \operatorname{Spec} A'$ as in the proposition statement. Then setting Z to be the preimage of Z in $\operatorname{Spec} A$ will have the desired property. Indeed, given $\mathfrak{p} \in \operatorname{Spec} A$, let $\mathfrak{q} = \varphi(\mathfrak{p})$. Then for any field k extending $\kappa(\mathfrak{p})$, the homomorphism φ induces k as a field extension of $\kappa(\mathfrak{q})$ as well, and considering f over k by the given extension is the same as considering f' over k by the induced extension. Obviously, if f factors over every algebraically closed extension of $\kappa(\mathfrak{p})$, it does so over some algebraically closed extension, and similarly for having degree less than d. So, suppose we have some algebraically closed k extending $\kappa(\mathfrak{p})$, with f either reducible or of degree less than d over k. Then the same applies to f' when considering k over $\kappa(\mathfrak{q})$, so we conclude that $\mathfrak{q} \in Z'$, and hence that $\mathfrak{p} \in Z$. Finally, if $\mathfrak{p} \in Z$, let k be any algebraically closed extension field of $\kappa(\mathfrak{p})$; then considering k over $\kappa(\mathfrak{q})$ we have by hypothesis that f' is reducible or has degree less than d over k, so the same is true for f, as desired.

This reduces the problem to the universal case, where $A = A_{d,n} := \mathbb{Z}[\{y_\alpha\}_{\alpha \in S(d,n)}]$, where $S(d,n)$ is the set of nonnegative vectors in $\mathbb{Z}^n$ with sum at most d, and $f = \sum_\alpha y_\alpha \vec{x}^\alpha$. In general, we see that for any ring B, polynomials of degree at most d in n variables with coefficients in B correspond precisely to morphisms $\operatorname{Spec} B \to \operatorname{Spec} A_{d,n}$. The idea is then quite simple: for each pair d_1, d_2 of positive integers with $d_1 + d_2 = d$, polynomial multiplication is expressed by polynomials which are homogeneous (indeed, linear) in the coefficients of each multiplicand. We thus get an induced morphism of affine spaces

$$\mu_{d_1,d_2} : \operatorname{Spec} A_{d_1,n} \times \operatorname{Spec} A_{d_2,n} \to \operatorname{Spec} A_{d,n},$$

with the property that a polynomial g with coefficients in any ring B, expressed as a morphism $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A_{d,n}$, can be written as a product of polynomials in B having degrees at most d_1 and d_2 if and only if φ factors through μ_{d_1,d_2}.

Next, by homogeneity and the observation that a product of two polynomials is zero if and only if one of the polynomials is zero, Lemma 2.6 implies that if we take the union over pairs (d_1, d_2) of the images of the μ_{d_1,d_2}, we get a closed subset. We claim this is our desired Z. Given $\mathfrak{p} \in \operatorname{Spec} A$, suppose that $\mathfrak{p} \in Z$. Then for some (d_1, d_2), we have $\mathfrak{p}$ in the image of μ_{d_1,d_2}. According to Lemma 2.5, if k is any algebraically closed extension of $\kappa(\mathfrak{p})$, the morphism $\operatorname{Spec} k \to \operatorname{Spec} A_{d,n}$ which corresponds to considering f over k must factor through μ_{d_1,d_2}. According to the property of μ_{d_1,d_2}, we conclude that when f is considered over k, it can be written as a product $f_1 f_2$, where f_1 and f_2 have degree at most d_1, d_2 respectively. Thus, over k either f is reducible, or it has degree less than d. We conclude that for our choice of Z, we have that (1) implies (2). Of course, (2) implies (3), so finally, suppose we are given some algebraically closed k extending $\kappa(\mathfrak{p})$ such that f is either reducible or has degree strictly less than d. In the former case, choose (d_1, d_2) such that the factors of f have degree at most d_1 and d_2; then these factors induce a morphism

$\operatorname{Spec} k \to \operatorname{Spec} A_{d_1,n} \times \operatorname{Spec} A_{d_2,n}$, which when composed with μ_{d_1,d_2} gives the morphism realizing f over k, so the composition has image $\mathfrak{p}$, and we conclude $\mathfrak{p} \in Z$. If the degree of f over k is less than d, then it may be written as a constant times a polynomial of degree at most $d-1$, so choosing $(d_1, d_2) = (1, d-1)$ and arguing as above, we see again that $\mathfrak{p} \in Z$. Thus, (3) implies (1), and the proposition is proved. $\square$

3. Curves in varieties

We now give the promised application to curves in varieties, or more generally schemes of finite type over a field. We begin with an easy lemma.

Lemma 3.1. *Let X be a scheme of finite type over a field, $Z \subseteq X$ a closed subscheme which does not contain any irreducible component of X, and $x \in Z$ a closed point. Then there exists an integral closed subscheme $Z' \subseteq X$ which contains x, has dimension 1, and is not contained in Z.*

Proof. Observe that integral closed subschemes correspond bijectively to closed subsets, so the statement is topological in nature. Without loss of generality, we may restrict to any irreducible component of X containing x, and therefore assume X is irreducible. Because X is of finite type over a field, we may further restrict to an affine neighborhood of x without affecting dimensions. We then argue by induction on $d := \dim X$. If $d = 1$, then Z must consist of finitely many points, and we can take Z to be (the reduced structure) on X itself. If $d > 1$, let $\eta_1, \ldots, \eta_m$ be the generic points of irreducible components of Z containing x and having dimension $d-1$ (we may have $m = 0$). Then, if $X = \operatorname{Spec} A$, and $\mathfrak{p}$ is the maximal ideal corresponding to x, and $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ are the prime ideals corresponding to $\eta_1, \ldots, \eta_m$, we have that each $\mathfrak{q}_i$ is strictly contained in $\mathfrak{p}$. Since a prime ideal cannot be a finite union of prime ideals, there is some $f \in A$ such that $f \in \mathfrak{p}$ but $f \notin \mathfrak{q}_i$ for any i. Let X' an irreducible component of the vanishing set of f which contains x. Then $\dim X' = d-1$, and by construction X' is not contained in Z, so we conclude the desired result by induction. $\square$

The main theorem is then the following.

Theorem 3.2. *Let X be a connected, positive-dimensional scheme of finite type over a field k. Then for any closed points $x, x' \in X$, there exists a connected closed subscheme $Z \subseteq X$ containing x and x' and having dimension 1.*

With the tools now at our disposal, the proof is not difficult, and comes down to basic field theory. The technique we use is both powerful and common in algebraic geometry: fibering X over a curve.

Proof. The proof is by induction on $\dim X$. If $\dim X = 1$, there is nothing to prove. If $\dim X = d > 1$, we see by an easy induction on the number of components that we may assume X is irreducible. Let $K(X)$ be the residue field of X at the generic point. It follows from Lemma 3.1 that if the desired statement holds for a (dense) open subset of X, then it holds for X, so we may restrict X to open subsets whenever necessary. Now, let $t \in K(X)$ be any element transcendental over k, and let K be the separable algebraic closure of $k(t)$ inside $K(X)$. The main claim is that $K/k(t)$ is a finite extension.

Observe that any algebraic field extension which is not finite contains subextensions of arbitrarily large degree. The primitive element theorem then implies that if the extension is separable, it contains elements of arbitrarily large degree. Now, $K/k(t)$ is separable by construction. On the other hand, if we extend t to a transcendence basis $t_1 = t, t_2, \ldots, t_d$ of $K(X)$ over k, it is not too difficult to check that for any $\alpha \in K(X)$ which is algebraic over $k(t)$, its minimal polynomial over $k(t)$ remains irreducible over $k(t_1, \ldots, t_d)$, so its degree over $k(t)$ is the same as its degree over $k(t_1, \ldots, t_d)$. But $K(X)$, being algebraic and finitely generated over $k(t_1, \ldots, t_d)$, is finite, so its elements have bounded degree over $k(t_1, \ldots, t_d)$, and we conclude that those elements which are algebraic over $k(t)$ likewise have bounded degree over $k(t)$. This proves that K is finite over $k(t)$, as claimed.

We thus conclude that $K/k(t)$ is finite, and in particular K is finitely generated over k. Thus, there exists an affine integral scheme Y of finite type with $K(Y) = K$. Since K has transcendence degree 1 over k, we have $\dim Y = 1$.

The inclusion $K \hookrightarrow K(X)$ induces a dominant rational map $X \dashrightarrow Y$, and replacing X by a suitable open subset, we may assume we have a dominant morphism $f : X \to Y$. We have $K(Y)$ separably closed in $K(X)$ by construction, and X being irreducible implies that the generic fiber of f is irreducible, so we conclude that the generic fiber is geometrically irreducible. By Theorem 2.2, it follows that by restricting to an open subset of Y, we will have that every fiber of f is geometrically irreducible, and in particular irreducible. Now, for any closed point $x \in X$, since X is irreducible and dominates Y, the fiber over $f(x)$ (which is necessarily a closed point of Y) cannot contain an irreducible component of X, and must in particular have dimension strictly less than d. Thus if we fix such an x, Lemma 3.1 implies that there is some Z an integral closed subscheme of X of dimension 1 containing x, and not contained in the fiber over $f(x)$. We then have that Z dominates Y, and in particular, $f(Z)$ must be an open subset of Y. Again restricting to this open subset, we may suppose that Z surjects onto Y. Now, since fibers over closed points are smaller-dimensional, by induction any two closed points in a given fiber lie in some connected closed subscheme of dimension 1. But Z connects any two fibers, so we obtain the desired result. □

Remark 3.3. Although we made extensive use of the scheme point of view in the results leading up to Theorem 3.2, most notably in the statement of Theorem 2.2, these results should not be viewed as applications of scheme theory. Indeed, the main points were already well understood in the classical context; see for instance §II.6 of [Sha94] for a development in the context of language of varieties. However, we may view scheme theory as providing an important pedagogical contribution, in that the statement of Theorem 2.2 is far more natural from the scheme point of view than when expressed in terms of algebraic closures of one function field in another.

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