PROPERTIES OF PROPERTIES OF MORPHISMS

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There are a dizzying array of properties of morphisms of schemes. Just in chapters II and III of Hartshorne, the following properties are defined: locally of finite type, of finite type, finite, open immersion, closed immersion, quasi-compact, quasi-finite, dominant, generically finite, separated, closed, universally closed, proper, projective, quasi-projective, immersion, affine, birational, flat, etale, smooth, and unramified. This omits additional properties of morphisms such as open and universally open, locally of finite presentation and of finite presentation, quasi-affine, radicial, quasi-separated, and monomorphism, among others. The purpose of this note is to help you organize this flood of definitions by familiarizing yourself with some common behaviors of properties of morphisms.

1. Local properties

Some properties have "locally" in their names, while others, such as flatness and smoothness, are local but do not include this in their names because there is no global version of them. We begin by discussing what it means to be local.

Notation 1.1. To have consistent notation, we set the following: if $f : X \to Y$ is a morphism, and $U \subseteq X$ open, then $f|_U$ denotes the induced morphism $U \to Y$. If $V \subseteq Y$ is open, then $f|_V$ denotes the induced morphism $f^{-1}(V) \to V$.

Definition 1.2. We say a property P of morphisms is **local on the source** if, given any morphism $f : X \to Y$, and any open covering $\{U_i\}$ of X, then f has P if and only if

$$f|_{U_i} : U_i \to Y$$

has P for each i.

Equivalently, P is local on the source if P is preserved by restriction to open subsets of X, and if P may be checked on an open cover of X.

We say a property P of morphisms if **local on the target** if, given any morphism $f : X \to Y$, and any open covering $\{V_i\}$ of Y, then f has P if and only if

$$f|_{V_i} : f^{-1}(V_i) \to V_i$$

has P for each i.

Equivalently, P is local on the target if P is preserved by restriction to open subsets of Y, and if P may be checked on an open cover of Y.

The idea is that nearly all properties of morphisms should be local on the target, while being local on the source is less common. Thus, typically when we say that a property is local, we mean on the source. Under mild hypotheses which we always hope should be satisfied, we see that being local on the source is stronger than being local on the target.

Proposition 1.3. *Suppose that a property P of morphisms satisfies the following:*

(i) *isomorphisms have P;*
(ii) *P is closed under composition;*
(iii) *P is closed under restriction to (the preimage of) open subsets of the target.*

Then if P is local on the source, it is also local on the target.

Observe that condition (iii) is a special (and typically easy) case of P being preserved under base change. In particular, nearly every property of morphisms satisfies (i)-(iii).

Proof. The first observation is that since P is local on the source, and open immersions are compositions of isomorphisms with restriction to an open subset of the source, we see that open immersions have P. Next, since we assume that restriction to an open subset of the target preserves P, it is enough to verify that P may be checked on open subsets of the target. So, suppose we have $f : X \to Y$ and an open cover $\{V_i\}$ of Y such that

$$f|_{V_i} : f^{-1}(V_i) \to V_i$$

has P for each i. Composing with $V_i \to Y$, we conclude that

$$f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to Y$$

has P for each i. Since P is local on the source, we conclude that f has P. Thus, P is local on the target. □

We now state some properties of morphisms which are local on the source and on the target; for future reference, we include some properties which we have not yet covered. We omit proofs, although in many cases the localness is immediate from the definition.

Proposition 1.4. *The following properties of morphisms are local on the source and on the target:*

- *locally of finite type;*
- *locally of finite presentation;*
- *flat;*
- *smooth;*
- *etale;*
- *unramified.*

Proposition 1.5. *The following properties of morphisms are local only on the target:*

- *quasi-compact;*
- *of finite type;*
- *open immersion;*
- *closed immersion;*
- *immersion;*
- *finite;*
- *quasi-finite.*

These lists are not comprehensive, but note that we have intentionally omitted projective and quasi-projective, as they are not local even on the target.

2. Affine communication

A number of properties of morphisms are defined in terms of the existence of an affine open cover of the target satisfying a certain property. One then wants to show that it follows that an arbitrary affine open subset of the target satisfies the same property. Following Vakil, a useful tool for accomplishing this is the "affine communication lemma." To avoid ambiguity, we use the notation X_f for the distinguished open subset of an affine scheme X associated to $f \in \mathcal{O}_X(X)$.

Lemma 2.1. *Let X be a scheme, and let P be a property of affine open subschemes of X such that:*

(i) *if $U = \operatorname{Spec} A \subseteq X$ has property P, then U_f has property P for all $f \in A$;*
(ii) *if $U = \operatorname{Spec} A \subseteq X$, and we have $f_1, \dots, f_n \in A$ such that the U_{f_i} cover U, and if U_{f_i} has property P for all i, then U has property P.*

Under these conditions, if there exists a cover $\{U_i\}$ of X by affine open subschemes with property P, then every affine open subset U of X has property P.

Recall that the U_{f_i} cover U if and only if the f_i generate the unit ideal in A.

The main statements we need for the lemma are the following:

Proposition 2.2. *Given X an affine scheme, $f \in \mathcal{O}_X(X)$, and $g \in \mathcal{O}_{X_f}(X_f)$, then $(X_f)_g = X_h$ for some $h \in \mathcal{O}_X(X)$.*

Proof. Since $\mathcal{O}_{X_f}(X_f) = \mathcal{O}_X(X_f) = (\mathcal{O}_X(X))_f$, we can write $g = \frac{g'}{f^m}$ for some $g' \in \mathcal{O}_X(X)$; then inside X_f, the vanishing set of g is the same as the vanishing set of g', so we see that we may set $h = fg'$. $\square$

Proposition 2.3. *Suppose that U, V are affine open subschemes of a scheme X. Then $U \cap V$ can be covered by affine open subschemes each of which is simultaneously a distinguished open subscheme of U and V.*

Proof. Since distinguished open subsets form a base of the topologies of both U and V, given $P \in U \cap V$ there exist $f \in \mathcal{O}_U(U)$ and $g \in \mathcal{O}_V(V)$ such that $U_f \subseteq U \cap V$, $V_g \subseteq U \cap V$, and $P \in U_f \cap V_g$. But we have that $U_f \cap V_g$ is equal to $(U_f)_{g|_{U_f}}$ and also to $(V_g)_{f|_{V_g}}$, so by Proposition 2.2 we conclude that it is a distinguished open subset of both U and V. $\square$

Proof of Lemma 2.1. By Proposition 2.3, there exists a cover U_{f_j} of U with the property that each U_{f_j} is also a distinguished open subset of (at least) one of the U_i, and thus by (i) we have that U_{f_j} has property P for all j. Since affine schemes are quasicompact, finitely many of the U_{f_j} suffice to cover U. Then in $\mathcal{O}_U(U)$ we have that the f_j generate the unit ideal, so by (ii) we conclude that U has property P, as desired. $\square$

We apply this to properties of morphisms as follows:

Corollary 2.4. *Let P be a property of morphisms from arbitrary schemes to affine schemes satisfying the following two conditions:*
if $f : X \to Y = \operatorname{Spec} B$ has property P, then for all $g \in B$,

$$f|_{Y_g} : f^{-1}(Y_g) \to Y_g$$

has property P;
given $f : X \to Y = \operatorname{Spec} B$, if we have $g_1, \ldots, g_n \in B$ such that the Y_{g_i} cover Y, and if

$$f|_{Y_{g_i}} : f^{-1}(Y_{g_i}) \to Y_{g_i}$$

has property P for all i, then f has property P.

Define the property P' of morphisms of schemes by saying that a morphism $f : X \to Y$ has property P' if there exists an open affine cover $\{V_i\}$ of Y such that each restriction

$$f|_{V_i} : f^{-1}(V_i) \to V_i$$

has property P. Then if f has property P', it follows that for every affine open subset $V \subseteq Y$, the restriction

$$f|_V : f^{-1}(V) \to V$$

has property P.

This is then applied to prove equivalent definitions for quasi-compact morphisms, morphisms (locally) of finite type, finite morphisms, and affine morphisms, to name a few. Although in some cases the affine communication lemma is also applied to covers of the source, this is less common and in order to avoid clutter we omit a general statement of this form.

Remark 2.5. Note that under the hypotheses of Corollary 2.4, it certainly follows that the property P' is local on the target. However, the conclusion of the corollary is far stronger than just being local on the target.

3. Formal games

Nearly all properties of morphisms are stable under base change, and under composition. In addition, properties which are in some sense related to bounding the source relative to the target tend to be satisfied by closed immersions, and sometimes by all immersions. This leads to a standard set of deductions.

Proposition 3.1. *Suppose a property P of morphisms satisfies the following conditions:*

(i) *closed immersions have P;*
(ii) *P is closed under composition;*
(iii) *P is closed under base change.*

Then it also satisfies:

(iv) *A product of morphisms having P has P;*
(v) *if $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ has P and g is separated, then f has P;*
(vi) *if $f : X \to Y$ has P, then $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has P.*

Suppose further that P satisfies:

(i′) *immersions have P.*

Then P also satisfies:

(v′) *if $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ has P, then f has P.*

The argument is as in Exercise II.4.8 of [1], and we omit it.

Below are some properties of morphisms to which the above argument applies.

Proposition 3.2. *The following properties of morphisms satisfy (i′), (ii) and (iii) of Proposition 3.1.*

- *locally of finite type;*
- *immersions;*
- *separated.*

The following properties of morphisms satisfy (i)-(iii) of Proposition 3.1:

- *quasi-compact;*
- *quasi-finite;*
- *of finite type;*
- *closed immersions;*
- *finite;*
- *universally closed;*
- *proper;*
- *projective.*

We leave the proofs as exercise. Note however that for separatedness and properness, one should argue directly from the definitions rather than via the valuative criteria, to avoid unnecessary hypotheses.

Note that closed immersions need not be locally of finite presentation. However, it is nonetheless true that:

Proposition 3.3. *The condition of being locally of finite presentation satisfies (ii)-(vi), and a modified version of (v) which requires g to be locally of finite type rather than separated.*

Note however that properties such as open immersion do not satisfy (v) in any form.

Remark 3.4. In fact, there is an intermediate condition between closed immersions having P and all immersions having P: that quasi-compact immersions have P. If one works out the resulting statements, it is possible to prove that quasi-compact (and hence finite type and quasi-finite) morphisms satisfy (v′) under mild hypotheses such as Y being locally Noetherian.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.