VALUATIVE CRITERIA

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Intuitively, one can think of separatedness as (a relative version of) uniqueness of limits, and properness as (a relative version of) existence of (unique) limits. It is not immediately obvious how to formalize these ideas in algebraic geometry, but it turns out to be doable, via valuative criteria. Aside from providing some intuition for separatedness and properness in terms of uniqueness and existence of limits, the valuative criteria are extremely important when working with moduli spaces. In this case they amount to studying the behavior of families of objects over valuation rings. We include the valuative criterion for universal closedness separately, because most algebraic stacks are not separated, and it is nonetheless helpful to know whether they are universally closed.

1. Statements

The most classical version of this sort of criterion applies to prevarieties over an algebraically closed field k. Let X be a prevariety. Then one can think of the setup for limits as follows: let C be a smooth curve over k, and $P \in C$ a point; then consider a morphism $f : C \smallsetminus \{P\} \to X$. We could picture that the points $f(Q)$ for $Q \in C \smallsetminus \{P\}$ have a limit as Q approaches P if f extends to a morphism on all of C. Uniqueness of the limit then corresponds to uniqueness of the extension. In fact, this works with prevarieties: X is a variety if and only if "limits are unique" in this sense, which is to say if and only if for all C, P and f as above, there is at most one extension of f to all of C. Similarly a variety X is complete if and only if "limits exist," which is to say for all C, P and f as above there exists a (necessarily unique) extension of f to all of C.

The above criterion works exactly the same if we replace C by the local scheme $\operatorname{Spec} \mathcal{O}_{C,P}$ and $C \smallsetminus P$ by $\operatorname{Spec} K(C)$, where $K(C)$ is the function field of C (and the field of fractions of $\mathcal{O}_{C,P}$). Now, $\mathcal{O}_{C,P}$ is a discrete valuation ring, and if we want to work with schemes not necessarily of finite type over an algebraically closed field, we should consider arbitrary discrete valuation rings. In fact, if we want to work with non-Noetherian schemes, we should consider not just discrete valuation rings, but arbitrary valuation rings. This is precisely what the valuative criteria do.

We begin by recalling the basic definitions and properties for valuation rings.

Definition 1.1. An integral domain A with fraction field K is a **valuation ring** if for all $x \in K^*$, either $x \in A$ or $x^{-1} \in A$.

Remark 1.2. The reason for the terminology is that we obtain a homomorphism ν from K^* to an ordered abelian group, with the property that $x \in A$ if and only if $\nu(x) \geqslant 0$. This homomorphism is simply obtained by setting the abelian group equal to K^*/A^*, with ordering determined by the above condition. Moreover, given $x, x' \in K^*$, with $x + x' \neq 0$, we have $\nu(x+x') \geqslant \min\{\nu(x), \nu(x')\}$. Indeed, suppose without loss of generality that $\nu(x) \leqslant \nu(x')$, so that by definition $x'/x \in A$. Then $(x + x')/x = 1 + x'/x \in A$, so $\nu(x + x') \geqslant \nu(x)$, as desired. Such a homomorphism is called a valuation, and as in §I.6 of Hartshorne [1], valuation rings may be defined equivalently in terms of existence of a valuation.

The topological space underlying a valuation ring may have arbitrary dimension, but we nonetheless have the following fact, which follows easily from Remark 1.2:

Proposition 1.3. *A valuation ring is a local ring.*

In particular, if A is a valuation ring, then $\operatorname{Spec} A$ has unique generic and closed points, corresponding to the zero ideal and the maximal ideal, respectively. Throughout our discussion, a morphism $\operatorname{Spec} K \to \operatorname{Spec} A$ is always assumed to be the canonical inclusion of the generic point.

We can now state the valuative criteria.

Definition 1.4. We say a morphism $f : X \to Y$ satisfies the **existence** (respectively, **uniqueness**) **part of the valuative criterion** if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with A a valuation ring and K its fraction field, there exists (respectively, there is at most one) one way of filling in the dashed arrow so that the diagram remains commutative.

Theorem 1.5. *Let $f : X \to Y$ be a morphism of schemes, and assume that Δ_f is quasicompact. Then f is separated if and only if it satisfies the uniqueness part of the valuative criterion.*

Theorem 1.6. *Let $f : X \to Y$ be a morphism of schemes, and assume that f is quasicompact. Then f is universally closed if and only if it satisfies the existence part of the valuative criterion.*

From the above two theorems, we immediately conclude the valuative criterion for properness.

Theorem 1.7. *Let $f : X \to Y$ be a morphism of finite type, with Δ_f quasicompact. Then f is proper if and only if it satisfies both the existence and uniqueness parts of the valuative criterion.*

Remark 1.8. Note that in the valuative criteria, if we have Y locally Noetherian and f locally of finite type, then it is enough to only consider discrete valuation rings instead of arbitrary ones. In addition, in this case (and more generally if X is locally Noetherian) the condition that Δ_f be quasi-comapct is automatically satisfied. Thus, under mild Noetherian hypotheses we do not need to worry about checking this condition separately.

More generally, the condition that Δ_f is quasi-compact is an annoying one, but it comes up naturally in a number of settings, enough so that it has a name – **quasi-separated**. Note that a separated morphism is necessarily quasi-separated, since closed immersions are quasi-compact. Thus, what the valuative criterion is really saying is that separatedness is the union of two properties: quasi-separatedness, and the valuative condition. Similarly, as we mentioned earlier, every universally closed morphism is quasi-compact, so the valuative criterion says that being universally closed is equivalent to being both quasi-compact and satisfying the valuative condition.

2. Proofs

Each valuative criterion is of course two statements: first, that separatedness (respectively, universal closedness) implies the stated criterion, and second, that the criterion implies separatedness (respectively, universal closedness). We thus have four statements to prove, and the proofs are rather independent of one another. There is however substantial commonality between the proofs that the two criteria imply separatedness and properness, so we will begin with the proofs of these statements.

The first concept we need is that of specialization:

Definition 2.1. Given a scheme X, and points $x, x' \in X$, we say that x **specializes** to x' if x' is in the closure of $\{x\}$. A subset $Z \subseteq X$ is **closed under specialization** if for all points $x, x' \in X$ with $x \in Z$ and x specializing to x', we also have $x' \in Z$.

Because schemes have (unique) generic points for all their irreducible closed subsets, the idea is that a subset of a scheme which is reasonably well behaved and closed under specialization should be closed. We then show that specializations (in both the relative and absolute settings) can be detected via morphisms from the spectra of valuation rings. Putting these statements together will quickly yield the relevant direction of the valuative criteria.

We begin by making the statement on closed sets and specialization more precise in an important special case:

Proposition 2.2. *Suppose $f : X \to Y$ is a quasi-compact morphism. If $f(X)$ is closed under specialization, then $f(X)$ is closed.*

For the proof, see Lemma 4.5 of Chapter II of Hartshorne [1].

Example 2.3. To see that quasicompactness is necessary, let $Y = \mathbb{A}^1_k$, and let X be an infinite disjoint union of closed points of $\mathbb{A}^1_k$, with f the inclusion. Then $f(X)$ contains only closed points, so is closed under specialization, but is not a closed set. Note that this morphism is even locally of finite type, so quasicompactness is really the crucial hypothesis.

Because f is not closed, this is also an example that the criterion of Theorem 1.6 does not imply that a morphism is closed without a quasicompactness hypothesis.

Proposition 2.2 will be enough for checking separatedness, but for universal closedness it is convenient to develop the statement into one on closed morphisms:

Corollary 2.4. *If $f : X \to Y$ is a quasicompact morphism, and for all $x \in X$, and $y \in Y$ such that $f(x)$ specializes to y, we have some $x' \in X$ with x specializing to x', and $f(x') = y$, then f is closed.*

Proof. Given $Z \subseteq X$ closed, give Z the reduced induced structure. Then since closed immersions are quasicompact, $Z \to Y$ is quasicompact, by the hypotheses we have the image of Z is closed under specialization, so $f(Z) \subseteq Y$ is closed by Proposition 2.2. □

The main use of valuation rings will be the following result, stating that specializations (in a relative and absolute setting) can be detected via morphisms from the spectra of valuation rings.

Proposition 2.5. *Let X be a scheme, and $x, x' \in X$ with x specializing to x'. Then there exists a valuation ring A and a morphism $\operatorname{Spec} A \to X$ with the generic point of $\operatorname{Spec} A$ mapping to x, and the closed point of $\operatorname{Spec} A$ mapping to x'.*

More generally, if $f : X \to Y$ is any morphism, and we have $x \in X$, and $y' \in Y$ a specialization of $f(x)$, then there exists a valuation ring A, with fraction field K, and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

such that the image of $\operatorname{Spec} K$ is x, and the generic and closed points of $\operatorname{Spec} A$ map to $f(x)$ and y respectively.

We defer the proof to the next section.

From this, it is not hard to prove that the stated valuative criteria imply separatedness and universal closedness.

Proof of "if" direction of Theorem 1.5. Suppose that the stated criterion is satisfied, so we wish to show that f is separated. Since we have assumed Δ_f quasi-compact, by Proposition 2.2 it is enough to show that the image of Δ_f is closed under specialization. Accordingly, suppose we have

z specializing to z' in $X \times_Y X$, with $z = \Delta_f(x)$ for some $x \in X$. By Proposition 2.5 there exists a valuation ring A with fraction field K, and a morphism $\psi : \operatorname{Spec} A \to X \times_Y X$ such that the generic point of $\operatorname{Spec} A$ maps to z, and the closed point of $\operatorname{Spec} A$ maps to z'.

Taking first and second projection yields two morphisms $p_1 \circ \psi$ and $p_2 \circ \psi$ from $\operatorname{Spec} A$ to X, which give the same morphism $\operatorname{Spec} A \to Y$ after composition with f. We claim that $p_1 \circ \psi$ agrees with $p_2 \circ \psi$ if we precompose with $\iota : \operatorname{Spec} K \to \operatorname{Spec} A$. It suffices to check that $\psi \circ \iota$ factors through Δ_f, but since this is a morphism from $\operatorname{Spec} K$, it is enough to observe that $\operatorname{Spec} K$ maps to z, which by hypothesis is a point of Δ_f. We thus obtain the claim, and then by hypothesis we conclude that $p_1 \circ \psi = p_2 \circ \psi$, and thus that ψ factors through Δ_f. It follows finally that $z' \in \Delta_f(X)$, so $\Delta_f(X)$ is closed under specialization, as desired. $\square$

The following lemma is used in checking properness. We leave the proof, which uses only the universal property of fibered products, to the reader.

Lemma 2.6. *Suppose a morphism $f : X \to Y$ satisfies the existence part of the valuative criterion. Then for every morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ of f satisfies the existence part of the valuative criterion.*

Proof of "if" direction of Theorem 1.6. Suppose our criterion is satisfied. Let $Y' \to Y$ be any morphism, and $X' = X \times_Y Y'$;. We thus wish to show that $X' \to Y'$ is closed. Since quasicompactness is preserved under base change, by Corollary 2.4 it is enough to show that for any $x \in X'$, and $y \in Y$ with $f(x)$ specializing to y, there exists $x' \in X'$ with x specializing to x' and $f(x') = y$.

By Proposition 2.5 there exists a valuation ring A with fraction field K and a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y' \end{tikzcd}$$

with the image of $\operatorname{Spec} K$ being x, and the image of the generic and closed points of $\operatorname{Spec} A$ being $f(x)$ and y, respectively. By Lemma 2.6, our criterion holds also for $X' \to Y'$, so we conclude that we can fill in the dashed arrow so that the diagram still commutes, and setting x' to be the image of the closed point of $\operatorname{Spec} A$ completes the argument. $\square$

We now move on to checking that conversely, separatedness and universal closedness also imply the stated criteria. We can prove the former directly, while the latter requires an additional result on morphisms and spectra of valuation rings.

Proof of "only if" direction of Theorem 1.5. Suppose that f is separated, and we have a diagram as in the statement, and morphisms $g_1, g_2 : \operatorname{Spec} \to X$ making the diagram commute. We thus obtain a morphism $g : \operatorname{Spec} A \to X \times_Y X$ such that $g \circ \iota$ factors through Δ_f, where $\iota : \operatorname{Spec} K \to \operatorname{Spec} A$ is the canonical inclusion. Because $\Delta_f(X)$ is closed by hypothesis, we conclude that $g(\operatorname{Spec} A) \subseteq \Delta_f(X)$, and thus that g factors through Δ_f, since $\operatorname{Spec} A$ is reduced. It thus follows that $g_1 = g_2$. $\square$

Proposition 2.7. *Let A be a valuation ring with fraction field K. Suppose that $f : X \to \operatorname{Spec} A$ is a closed morphism. Then given a morphism $\operatorname{Spec} K \to X$ such that the diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \operatorname{Spec} A \end{tikzcd}$$

commutes, there exists a morphism filling in the dashed arrow so that the diagram still commutes.

One can rephrase the proposition as saying that for a closed morphism to $\operatorname{Spec} A$, every generic section extends to a section. The proof of the proposition is deferred to the next section, but we can now easily conclude the proof of Theorem 1.6.

Proof of "only if" direction of Theorem 1.6. Suppose that f is universally closed. Given a diagram as in the valuative criterion, we wish to prove existence of the dashed arrow. Consider the base change $X' := X \times_Y \operatorname{Spec} A \to \operatorname{Spec} A$, which is closed by hypothesis. The morphism $\operatorname{Spec} K \to X$ then induces a morphism $\operatorname{Spec} K \to X'$, and by Proposition 2.7 we obtain a morphism $\operatorname{Spec} A \to X'$ which when composed with the projection morphism $X' \to X$, gives us what we want. □

3. Proofs of statements on valuation rings

Recall that if K is a field, we have a partial order on the local rings strictly contained in K given by $A \leqslant B$ if $A \subseteq B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$, or equivalently if the inclusion $A \to B$ is a local homomorphism. This partial order is called **dominance**. The basic algebra fact we need is the following:

Proposition 3.1. *The valuation rings A with given fraction field K are precisely the maximal elements under dominance among local rings strictly contained in K.*

The proof of this is not difficult; see Theorem 10.2 of Matsumura [2].

Proof of Proposition 2.5. Note that the first statement follows from the second, by taking f to be the identity map. For the second statement, by replacing Y with the closure of $f(x)$ (with the reduced induced subscheme structure), it is clear that we may assume Y is integral, and $f(x)$ is the generic point of Y. Then, by Proposition 3.1, there exists a valuation ring A' of $K(Y)$ dominating $\mathcal{O}_{Y,y}$, so we have a morphism $\operatorname{Spec} A' \to Y$ sending the generic point to $f(x)$ and the closed point to y. Since $K(Y) = k(f(x))$, we also have an inclusion $K(Y) \hookrightarrow k(x)$, so we can let A be a valuation ring of $k(x)$ which dominates A'. Then we obtain a morphism $\operatorname{Spec} A \to Y$ which sends the generic point to $f(x)$ and the closed point to y, and by construction the fraction field of A is $k(x)$, so there is a canonical morphism from $\operatorname{Spec} K := \operatorname{Spec} k(x) \to X$ with image x commuting with the given morphisms, as desired. □

Note that the proof in fact shows that we can always assume $\operatorname{Spec} A \to X$ induces an isomorphism $K \cong k(x)$.

Proof of Proposition 2.7. Let x be the image of $\operatorname{Spec} K$; then by hypothesis, $f(x)$ is the generic point of $\operatorname{Spec} A$. If $Z \subseteq X$ is the closure of x, the hypothesis that f is closed implies that $f(Z) = \operatorname{Spec} A$, so there is some $x' \in X$ with x specializing to x', and $f(x')$ being the closed point of $\operatorname{Spec} A$. The morphisms $\operatorname{Spec} K \to X \to \operatorname{Spec} A$ having composition equal to the canonical inclusion must then yield an isomorphism $k(x) \cong K$. Thus, if we put the reduced subscheme structure on Z, we have $Z \to \operatorname{Spec} A$ inducing an isomorphism on function fields. Then since x' maps to the closed point, we have that $\mathcal{O}_{Z,x'}$ dominates A, and must therefore be equal to A by Proposition 3.1. We conclude that the canonical morphism $\operatorname{Spec} \mathcal{O}_{Z,x'} \to Z \to X$ yields the desired morphism $\operatorname{Spec} A \to X$. □

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.