RECOVERING CLASSICAL NOTIONS OF POINTS AND MORPHISMS IN
THE SCHEME SETTING

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Having made the basic definitions of schemes, we now step back to investigate how classical
notions fit in. We have already discussed the scheme associated to a classical prevariety over
an algebraically closed field $k$, and we will begin by investigating how to recover morphisms of
prevarieties in the scheme setting. Dropping the algebraic closure hypothesis, we will also see how
to understand classical points on an affine variety over an arbitrary field $k$ in the scheme setting.
In both of these topics, we will see vividly Grothendieck’s shift from objects to morphisms as the
basic subject of interest.

1. Morphisms of prevarieties

Let $k$ be an algebraically closed field, and $X$ and $Y$ prevarieties over $k$. We have already
associated schemes $X'$ and $Y'$ to $X$ and $Y$, but we recall that prevariety morphisms $X \rightarrow Y$
do not correspond to scheme morphisms $X' \rightarrow Y'$.

Example 1.1. In the context of varieties, there is a unique morphism from a point to itself. In
contrast, we have seen that morphisms Spec $k \rightarrow \text{Spec } k$ correspond to field inclusions $k \hookrightarrow k$;
thus, they are not unique, and need not even be invertible!

At first, this looks very bad, but in fact, we will see that with the right idea, the situation is
eminently fixable. For simplicity, we begin our investigation with the case that $X$ and $Y$ are affine;
in this setting, we know that prevariety morphisms $X \rightarrow Y$ correspond to $k$-algebra homomorphisms
$A(Y) \rightarrow A(X)$. In contrast, scheme morphisms $X' = \text{Spec } A(X) \rightarrow Y' = \text{Spec } A(Y)$ correspond
to arbitrary ring homomorphisms $A(Y) \rightarrow A(X)$. Thus, what is missing in the scheme setting is
the $k$-algebra structure. The basic observation is then as follows: because a $k$-algebra structure on
a ring $R$ is uniquely determined by a homomorphism $k \rightarrow R$, the $k$-algebra structure on $A(X)$ is
recorded by a homomorphism $k \rightarrow A(X)$, which is the same as a scheme morphism $X' \rightarrow \text{Spec } k$,
and similarly for $Y$. Next, if $R$ and $S$ are $k$-algebras, a ring homomorphism $R \rightarrow S$ is a $k$-algebra
homomorphism if and only if it commutes with the two inclusions of $k$, so we see that a scheme
morphism $X' \rightarrow Y'$ corresponds to a $k$-algebra homomorphism $A(Y) \rightarrow A(X)$ if and only if it
commutes with the morphisms $X' \rightarrow \text{Spec } k$ and $Y' \rightarrow \text{Spec } k$.

The conclusion is that in order to remember that $X'$ and $Y'$ came from prevarieties over $k$,
we should not think of them just as schemes, but as schemes together with morphisms to $\text{Spec } k$.
In this context, prevariety morphisms $X \rightarrow Y$ correspond to morphisms $X' \rightarrow Y'$ “over $\text{Spec } k$.”
In fact, this pictures generalizes from the affine case to arbitrary prevarieties: according to your
homework, a morphism $X' \rightarrow \text{Spec } k$ is the same as a ring homomorphism $k \rightarrow \mathcal{O}_X'(X')$, and
because $\mathcal{O}_X'(X') = \mathcal{O}(X)$ is the ring of global regular functions on $X$, it comes with a $k$-algebra
structure. We have:

Exercise 1.2. If $X$ and $Y$ are prevarieties over $k$, and $X' \rightarrow \text{Spec } k$ and $Y' \rightarrow \text{Spec } k$ the corre-
sponding schemes with morphisms to $\text{Spec } k$, then prevariety morphisms $X \rightarrow Y$ correspond to
scheme morphisms $X' \rightarrow Y'$ which commute with the two morphisms to $\text{Spec } k$.

As one might expect, the setup of working with schemes “over a base scheme” becomes ubiqituous
in scheme theory, leading to the following definitions:
Definition 1.3. If $S$ is a scheme, an $S$-scheme $X$ is a scheme (also denoted $X$) together with a morphism $X \to S$ (called the structure morphism). A morphism of $S$-schemes $X \to Y$ is morphism of the underlying schemes which commutes with the two morphisms to $S$.

When $S = \text{Spec} R$ is affine, we often refer instead to $R$-schemes and their morphisms. Thus, we can rephrase Exercise 1.2 as saying that in order for our schemes $X'$ and $Y'$ to behave like the original varieties $X$ and $Y$, we need to think of them not as schemes, but as $k$-schemes.

This is the first indication of the centrality of morphisms in Grothendieck’s theory: since every classical variety is not just a scheme, but a $k$-scheme, it is natural that properties which were formerly considered to be properties of objects will now be transformed into properties of morphisms.

Remark 1.4. From your homework, we see that every scheme $X$ has a unique morphism to $\text{Spec} \mathbb{Z}$. Thus, considering schemes and their morphisms is the same as considering $\mathbb{Z}$-schemes and their morphisms. In particular, if we phrase a definition or result in terms of $S$-schemes, by setting $S = \text{Spec} \mathbb{Z}$ we recover as a special case the situation of schemes not over a base.

2. Points over arbitrary fields

In the case that $k$ is algebraically closed, we have seen that if $X$ is a prevariety over $k$, then the points of $X$ correspond to the closed points of $X'$. However, it turns out that this is a somewhat misleading description which does not generalize well. This underlines the philosophy that while varieties had the “right” points but the “wrong” topology (in the sense that we could not invoke the Zariski topology directly for meaningful definitions of, for instance, compactness), both the points and topology of schemes are “wrong,” in the sense that they are useful as a technical tool, but do not correspond to classical notions. To understand classical points in the scheme context, instead of thinking of points as elements of the underlying set of a variety, it will be more profitable to think of points as corresponding to morphisms from the one-point variety into $X$.

Indeed, suppose that $k$ is not algebraically closed. As we have mentioned previously, it is quite difficult to even develop a general theory of abstract varieties over $k$, but at least in the affine (and also projective) case it is clear what we want to study: an affine variety $X$ over $k$ in $k^n$ should be defined by polynomials $f_1, \ldots, f_m$ with coefficients in $k$, and the points of $X$ over $k$ should be $n$-tuples of elements of $k$ on which all the $f_i$ vanish. In the case that $k$ is not algebraically closed, it becomes important to also consider the points over various extensions of $k$, so more generally, if $k'$ is an extension of $k$, the points of $X$ over $k'$, denoted $X(k')$, consist simply of elements of $(k')^n$ on which all the $f_i$ vanish.

Now, let $I \subseteq k[x_1, \ldots, x_n]$ be the ideal generated by the $f_i$; we will rephrase the sets $X(k')$ algebraically, which then leads to a scheme-based description of them. Write $A(X) = k[x_1, \ldots, x_n]/I$. We have seen that the underlying set of Spec $A(X)$ may not have much to do with the points of $X$ over any particular field. However, a point $(c_1, \ldots, c_n) \in (k')^n$ gives a homomorphism $\varphi : k[x_1, \ldots, x_n] \to k'$ by sending each $x_i$ to $c_i$; moreover, every homomorphism arises in this way precisely when we add the further restriction that they commute with the inclusions $k \to k[x_1, \ldots, x_n]$ and $k \to k'$. Further, we see that $(c_1, \ldots, c_n) \in X(k')$ if and only if $I$ is in the kernel of $\varphi$, or equivalently, if and only if $\varphi$ induces a homomorphism $A(X) \to k'$. Putting this together, we conclude that $X(k')$ corresponds to homomorphisms $A(X) \to k'$ commuting with the inclusions $k \to A(X)$ and $k \to k'$. Translating this into the context of affine schemes, we conclude:

Proposition 2.1. For any $k'$ extending $k$, consider Spec $k'$ and Spec $A(X)$ as $k$-schemes. Then the set $X(k')$ corresponds to $k$-scheme morphisms Spec $k' \to$ Spec $A(X)$.

Thus, using the notion of $k$-scheme, we can transparently describe points of $X$ over any $k'$ in terms of maps from a “one-point variety” into $X$, where scheme theory lets us naturally consider different one-point “varieties” depending on $k'$. Aside from its naturality and simplicity, the beauty of this
description is that it makes sense for any \( k \)-scheme, without any affineness hypothesis. Although we haven’t made enough definitions outside the affine setting to verify that this generalized scheme definition gives the right behavior, suffice it to say that for any notion of variety over a non-algebraically closed field, the definition of its points will agree with the definition suggested by Proposition 2.1. We do mention the following special case.

Exercise 2.2. If \( k \) is algebraically closed, and \( X \) is a prevariety over \( k \), with \( X' \) the associated \( k \)-scheme, then the points of \( X \) correspond to the \( k \)-scheme morphisms Spec \( k \to X' \).

In fact, in the above we haven’t used anything about \( k \) and \( k' \) being fields: if we have polynomials with coefficients in a ring \( R \), and \( S \) is an \( R \)-algebra, then it makes sense to talk about solutions over \( S \), and we see that we get exactly the same scheme-based description of these solutions. This comes up naturally for instance when \( R = \mathbb{Z} \) and \( S = \mathbb{F}_p \), so that we have integer equations and want to consider solutions modulo \( p \) for various \( p \). Indeed, from the point of view of Proposition 2.1, there is no reason to restrict ourselves to solutions over rings, so we make the following definition:

Definition 2.3. Suppose \( S \) is a scheme, and \( X \) is an \( S \)-scheme. Then for any \( S \)-scheme \( T \), the \( T \)-valued points of \( X \), denoted \( X(T) \) is the set of \( S \)-scheme morphisms \( T \to X \).

As before, if \( T = \text{Spec} R \) is affine, we will frequently say \( R \)-valued points instead of \( T \)-valued points. A \( T \)-valued point of a scheme \( X \) can be thought of as a family of points of \( X \), indexed by the points of \( T \). The concept of \( T \)-valued points will play a crucial role in Grothendieck’s ideas of the development of moduli spaces, to which we will return later in the quarter.

Remark 2.4. Yoneda’s lemma says that an \( S \)-scheme is uniquely determined by its \( T \)-valued points as \( T \) runs over all \( S \) schemes; less tautological is that it is actually determined by its \( R \)-valued points as \( R \) runs over all (rings corresponding to) affine \( S \)-schemes. So in some sense, we can think of a scheme as being given by its \( R \)-points, as \( R \) varies over all rings.