MODULI SPACES, REPRESENTABLE FUNCTORS, AND FIBERED PRODUCTS

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The idea of a moduli space is that its points correspond to a geometric object of interest. The underlying set of the space is thus determined, but not its geometric structure. In most cases, one can describe a geometric structure in an ad hoc manner, but it is not immediately obvious how to rigorously formulate the idea that the resulting object is “the” moduli space for the problem. This is the problem addressed by the theory of representable functors, with the key idea being that we specify not only the objects of interest, but also what families of these objects should look like.

1. Grassmannians

We begin by considering the example of a Grassmannian $G(r, d)$. The motivation part of the discussion will occur over a fixed algebraically closed field $k$, in the context of varieties. Then the space $G(r, d)$ is supposed to parametrize $r$-dimensional subspaces of a $d$-dimensional $k$-vector space $V$. Without saying anything more, this only specifies $G(r, d)$ as a set, and doesn’t say anything about its geometry – we don’t know anything about how the points are supposed to “fit together.” The basic idea is that instead of just specifying the geometric objects we are interested in (which correspond to points of the moduli space), we should also specify what families of these objects look like (which says something about how the points fit together).

In the case of a Grassmannian, what is a family of $r$-dimensional subspaces of a $d$-dimensional vector space? If the family is indexed by the points of a variety $T$, we will obtain a map $T \to G(r, d)$. The key idea is then the following: if we specify what “nice” families of subspaces should be, these should correspond to “nice” maps $T \to G(r, d)$, which is to say, morphisms. Thus, we will be describing what all morphisms $T \to G(r, d)$ will be for all varieties $T$, and this will, it turns out, describe the variety structure on $G(r, d)$. We can express this by saying that if we not only describe the set of $k$-valued points of $G(r, d)$, but also the set of $T$-valued points for all $T$, then $G(r, d)$ will be uniquely determined.

Now, the most natural way to describe nice families of vector spaces is in terms of subbundles of the trivial vector bundle. That is, a family of subspaces indexed by $T$ should be a rank-$r$ subbundle of the trivial vector bundle $A^d_T := A^d_k \times T$. The condition that we actually have a subbundle is ensuring that these subspaces fit together nicely into an algebraic family, and are not just some random collection of subspaces indexed by the points of $T$.

Formally, one ought to define a vector bundle and a subbundle as geometric objects. However, given the geometric objects, one can always take the sheaf of sections, and this gives an equivalent formulation in sheaf theory (see Exercise 5.18 of Chapter II of [1]). The two formulations are often used interchangeably, and the sheaf theoretic point of view will be easier for us to work with, so we make our definitions in that setting.

We now shift from varieties to schemes, and make our precise definitions in that context. Although they would also work in the context of varieties, and although the Grassmannian will ultimately be a variety (in fact, a nonsingular one), working systematically with schemes will give us more information – for instance, a description of the tangent space of the Grassmannian.
Definition 1.1. Given a scheme $T$, a vector bundle of rank $d$ on $T$ is a locally free sheaf $\mathcal{F}$ on $T$ of rank $d$. A subbundle of $\mathcal{F}$ of rank $r$ is a subsheaf $\mathcal{G}$ of $\mathcal{F}$ which is locally free of rank $r$, and such that $\mathcal{F}/\mathcal{G}$ is also locally free (necessarily of rank $d-r$).

Remark 1.2. In fact, if $\mathcal{F}/\mathcal{G}$ is locally free, it follows that $\mathcal{G}$ is locally free, but not conversely. We state the definition in this manner to emphasize the motivating geometric context. The reason for the necessity of assuming that $\mathcal{F}/\mathcal{G}$ is locally free is that an injective map of locally free sheaves does not necessarily correspond to an injective map of the corresponding geometric vector bundles: if the rank of the map drops on a closed subset of $T$, the map of sheaves is in general injective, but the map of geometric vector bundles is not. It turns out that the condition that $\mathcal{F}/\mathcal{G}$ be locally free is equivalent to injectivity of the associated map of geometric vector bundles, which is what should be required of a subbundle.

In particular, the condition that $\mathcal{F}/\mathcal{G}$ be locally free also ensures that the definition is invariant under pullback, so that if $\mathcal{G}$ is a subbundle of $\mathcal{F}$ on $T$, and $f : T' \to T$ any morphism, then $f^*\mathcal{G}$ is a subbundle of $f^*\mathcal{F}$.

We conclude that for the Grassmannian, the morphisms $T \to G(r, d)$ should correspond to rank-$r$ subbundles of the trivial vector bundle $\mathcal{O}_T^{\oplus d}$. But there is one additional point to address: when discussing varieties, it was clear how we obtained a family of subspaces from a morphism $T \to G(r, d)$, and vice versa, because everything was determined point by point. But for schemes, nonreduced structure makes this a bit subtler. The last ingredient is the idea of a universal family. Namely, if every morphism $T \to G(r, d)$ is supposed to correspond to some subbundle over $T$, then in particular the identity morphism $G(r, d) \to G(r, d)$ should correspond to a subbundle over $G(r, d)$ itself. This suggests a candidate for how to obtain the bijection from morphisms $T \to G(r, d)$ to subbundles on $T$: given a morphism, we simply pull back the universal subbundle. We thus have:

Definition 1.3. The Grassmannian $G(r, d)$ is a scheme over $\text{Spec} \, k$, together with a rank-$r$ subbundle $\mathcal{F}_{\text{univ}}$ of $\mathcal{O}_{G(r, d)}^{\oplus d}$, such that for every scheme $T$ over $\text{Spec} \, k$ and rank-$r$ subbundle $\mathcal{F}$ of $\mathcal{O}_T^{\oplus d}$, there exists a unique morphism $f : T \to G(r, d)$ over $\text{Spec} \, k$ such that $f^*\mathcal{F}_{\text{univ}}$ is identified with $\mathcal{F}$ under the canonical identification $f^*\mathcal{O}_{G(r, d)}^{\oplus d} = \mathcal{O}_T^{\oplus d}$.

We will see shortly that the Grassmannian is unique, if it exists. We will also describe techniques that can be used to prove that it exists.

Remark 1.4. Note that there is nothing special about $\text{Spec} \, k$ in the definition, other than that it gives us a variety over $\text{Spec} \, k$ in the end. We can replace it by any scheme $S$ to obtain a Grassmannian over $S$, or do away with it entirely (equivalently, set $S = \text{Spec} \, \mathbb{Z}$) to obtain a universal Grassmannian.

Similarly, we could replace the trivial vector bundle with any vector bundle $\mathcal{E}$ on $S$ of rank $d$, and then since the Grassmannian and the schemes $T$ are all over $S$, we can replace the trivial bundles in the definition with the pullbacks of $\mathcal{E}$ from $S$.

2. Representable functors

We now recast our discussion thus far in more abstract language. By associating to each $T$ the set of rank-$r$ subbundles of $\mathcal{O}_T^{\oplus d}$, we have constructed a (contravariant) functor $\mathcal{G}(r, d)$ from the category of schemes to the category of sets. Note that to get a functor, we need to know that subbundles are closed under pullback, which by Remark 1.2 is a consequence of the definition. On the other hand, for any scheme $X$, we have a functor $h_X$, the functor of points of $X$, which sends a scheme $T$ to the set of morphisms $T \to X$, with the functor structure on morphisms given by composition. Now, given any scheme $X$ and a rank-$r$ subbundle $\mathcal{F}$ of $\mathcal{O}_X^{\oplus d}$, we obtain a natural transformation of functors $h_X \to \mathcal{G}(r, d)$ given by sending $f : T \to X$ to $f^*\mathcal{F}$. Our definition of
the Grassmannian can be recast as saying that the morphism $h_{G(r,d)} \to G(r,d)$ given by $\mathcal{P}_{\text{univ}}$ is supposed to be an isomorphism of functors. This motivates the following:

**Definition 2.1.** Given a category $C$, and a contravariant functor $F : C \to \text{Set}$, suppose $X \in C$, and $\xi \in F(X)$. Then $(X, \xi)$ represents $F$ if the induced natural transformation $h_X \to F$ is an isomorphism.

Note that $h_X$ makes sense for any category. We can make a similar definition for covariant functors, but the contravariant one is of primary interest for moduli spaces. We could say instead that $X$ represents $F$ if there exists an isomorphism of functors $h_X \to F$, but it is helpful to specify the isomorphism. It is not hard to check that any isomorphism $h_X \to F$ is induced by an object of $F(X)$ as in the definition, so we have not imposed any undue restrictions.

**Example 2.2.** We showed that for any $X$, we have the morphisms $\varphi : X \to A^{1}_{Z}$ in bijection with $\mathcal{O}_{X}(X)$, via $\varphi \mapsto \varphi^{*}t$. We conclude that $(A^{1}_{Z}, t)$ represents the functor $X \mapsto \mathcal{O}_{X}(X)$.

Note that if we have a morphism $X \to Y$ in $C$, we obtain a morphism $h_{X} \to h_{Y}$ by composition, and the same holds for isomorphisms. Yoneda's lemma says, roughly speaking, that the condition of representing a given functor uniquely determines an object of a category. Specifically:

**Lemma 2.3.** (Yoneda's lemma) Given objects $X, Y \in C$, suppose we have a morphism $\varphi : h_{X} \to h_{Y}$ of the associated functor of points. Then $\varphi$ is induced by a unique morphism $f : X \to Y$. In particular, any isomorphism $h_{X} \xrightarrow{\sim} h_{Y}$ is induced by an isomorphism $X \xrightarrow{\sim} Y$.

**Proof.** We consider the map $\varphi(X) : h_{X}(X) \to h_{Y}(X)$. In $h_{X}(X)$ we have the identity element, and we set $f = (\varphi(X))(\text{id}) \in \text{Mor}(X, Y)$. It is then a simple exercise to check that this induces a bijection between morphisms $h_{X} \to h_{Y}$ and morphisms $X \to Y$. \qed

**Corollary 2.4.** Suppose that $(X, \xi)$ and $(X', \xi')$ both represent a functor $F$. Then there exists a unique isomorphism $X \to X'$ such that $\xi'|_{X} = \xi$.

In the above and throughout this note, we will use restriction notation to denote the map induced by a contravariant functor: that is, if $f : X \to X'$ is the given morphism, then $\xi'|_{X}$ is alternate notation for $F(f)(\xi')$.

**Proof.** Clearly we have $h_{X}$ isomorphic to $h_{X'}$, so by Yoneda’s lemma we have $X \cong X'$. However, the condition $\xi'|_{X} = \xi$ means that the isomorphism $h_{X} \xrightarrow{\sim} h_{X'}$ has to be the one induced by $h_{X} \xrightarrow{\xi} F \xleftarrow{\xi'} h_{X'}$, and Yoneda’s lemma says there is a unique isomorphism $X \to X'$ inducing this particular isomorphism. \qed

**Example 2.5.** We conclude that the Grassmannian is unique (up to unique isomorphism), if it exists.

Yoneda’s lemma is not only useful for moduli spaces, but also as a general principle for dealing with universal properties. In some cases (such as normalization) the phrasing can be a bit delicate, but a simple example is provided by the fibered product.

**Definition 2.6.** Given morphisms $f : X \to Z$ and $g : Y \to Z$ of schemes, the fibered product consists of a scheme $X \times_{Z} Y$ together with projection morphisms $p_{1}, p_{2}$ satisfying the following universal property: for any scheme $T$, and morphisms $q_{1} : T \to X$ and $q_{2} : T \to Y$ such that the induced morphisms to $Z$ agree, there exists a unique morphism $T \to X \times_{Z} Y$ yielding $q_{1}$ and $q_{2}$ after composing with $p_{1}$ and $p_{2}$.

The above definition makes sense in any category. For sets, we get the following very simple description.
Example 2.7. Recall that if \( X, Y, Z \) are sets and \( X \to Z, Y \to Z \) any functions, then \( X \times_Z Y \) is the subset of \( X \times Y \) consisting of pairs \((x, y)\) mapping to the same element in \( Z \).

We can easily rephrase the definition of fibered product in the language of functors: indeed, unwinding the definitions we are saying precisely that \((X \times_S Y, p_1, p_2)\) represents the functor \( X \times_Z Y \) defined by

\[
X \times_Z Y(T) = h_X(T) \times_{h_Z(T)} h_Y(T),
\]

where the fibered product on the right is simply set fibered product.

We conclude that the fact that the fibered product is uniquely determined by its universal property is a special case of Yoneda’s lemma.

3. Zariski sheaves

Next, we examine a certain property that a contravariant functor \( F : \text{Sch} \to \text{Set} \) must have in order to be representable, the property of being a “Zariski sheaf”. In fact, we will fix a base scheme \( S \), and work with the category \( \text{Sch}_S \) of schemes over \( S \). Recall that there is no loss of generality here, since if we want to work with the category \( \text{Sch} \) of all schemes, we can simply set \( S = \text{Spec} \mathbb{Z} \).

Recall the following statement on gluing morphisms:

**Proposition 3.1.** Let \( X \) and \( Y \) be schemes over \( S \), and \( \{U_i\} \) an open covering of \( X \). Then morphisms \( f : X \to Y \) over \( S \) are in one-to-one correspondence with collections of morphisms \( f_i : U_i \to Y \) over \( S \), such that for all \( i, j \) we have \( f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \) as morphisms \( U_i \cap U_j \to Y \) over \( S \).

This means that in order for a functor \( F : \text{Sch}_S \to \text{Set} \) to be representable, it needs to satisfy a certain property, analogous to the condition for a presheaf to be a sheaf:

**Definition 3.2.** A contravariant functor \( F : \text{Sch}_S \to \text{Set} \) is a Zariski sheaf if it satisfies the following condition:

For every \( X \in \text{Obj}(\text{Sch}_S) \), and every open cover \( \{U_i\} \) of \( X \), the natural map

\[
\{\eta \in F(X)\} \to \{\{\eta_i \in F(U_i)\} : \forall i, j, \eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}\}
\]

is a bijection.

It is simply a matter of definition-chasing to see that Proposition 3.1 can be rephrased to say the following:

**Corollary 3.3.** Let \( F : \text{Sch}_S \to \text{Set} \) be a contravariant functor. In order for \( F \) to be representable, it is necessary that \( F \) be a Zariski sheaf.

This is by no means a sufficient condition, but we will see that if \( F \) is a Zariski sheaf, and we want to construct an \( X \) representing \( F \), then it is enough to carry out the construction locally. We will use this to prove that fibered products exist in the category of schemes, and it can also be used to give a simple proof that the Grassmannian functor is representable.

As an example of a Zariski sheaf, let’s consider the fibered product.

**Proposition 3.4.** Given schemes \( X, Y, Z \) and morphisms \( f : X \to Z, \ g : Y \to Z \), the functor \( X \times_Z Y \) is a Zariski sheaf.

**Proof.** Indeed, this is just a formality from Proposition 3.1, since the functor is defined in terms of morphisms to \( X, Y \), and \( Z \). Specifically, let \( T \) be a scheme; by definition, \( X \times_Z Y(T) \) is defined to be the set of pairs of morphisms \( q_1 : T \to X, \ q_2 : T \to Y \) such that \( f \circ q_1 = g \circ q_2 \). If we have an open cover \( \{U_i\} \) of \( T \), because we know that \( h_X \) and \( h_Y \) are Zariski sheaves, we have that the morphism \( q_i : T \to X \) is uniquely determined by a collection of \( q_{1,i} : U_i \to X \) such that \( q_{1,i}|_{U_i \cap U_j} = q_{1,j}|_{U_i \cap U_j} \) for all \( i, j \), and similarly for \( q_2 : T \to Y \). Moreover, because morphisms \( T \to Z \) are uniquely
determined by collections of morphisms $U_i \to Z$, we also see that $f \circ q_1 = g \circ q_2$ if and only if $f \circ q_{1,i} = g \circ q_{2,i}$ for all $i$. Thus, an element of $X \times_Z Y(T)$ is determined uniquely by a family of elements of $X \times_Z Y(U_i)$ which agree on the $U_i \cap \overline{U}_j$, and we see that $X \times_Z Y$ is a Zariski sheaf. □

4. Representability of Zariski sheaves and the fibered product

We discussed that by definition, a fibered product $X \times_Z Y$ represents a functor, which we will denote by $X \times_Z Y$. Uniqueness of fibered products is true in any category, but existence is a question with much more substance. In particular, there are categories in which fibered products do not exist. We have seen that for schemes, the fibered product functor is a Zariski sheaf. We now observe that fibered products exist for affine schemes.

**Proposition 4.1.** Suppose that $X = \text{Spec} A$, $Y = \text{Spec} B$, $Z = \text{Spec} C$ are affine schemes. Then $X \times_Z Y$ is represented by $\text{Spec}(A \otimes_C B)$, with the natural maps to $\text{Spec} A$ and $\text{Spec} B$.

**Proof.** By your homework, we have $\text{Mor}(T, X) = \text{Hom}(A, \mathcal{O}_T(T))$ (where $\text{Hom}$ denotes the set of ring homomorphisms), and similarly $\text{Mor}(T, Y) = \text{Hom}(B, \mathcal{O}_T(T))$ and $\text{Mor}(T, \text{Spec}(A \otimes_C B)) = \text{Hom}(A \otimes_C B, \mathcal{O}_T(T))$. Also, the maps $\pi_X$ and $\pi_Y$ correspond to ring homomorphisms $\pi_X : C \to A$ and $\pi_Y : C \to B$ (which is what we use to define the tensor product). Let’s write $p_i^T : A \to A \otimes_C B$ and $\pi_{A,i} : B \to B \otimes_C B$ for the maps $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. We see that we want to check that composition with $\pi_X^i$ and $\pi_Y^i$ induces a bijection between $\text{Mor}(A \otimes_C B, \mathcal{O}_T(T))$ and $\{(f^i, g^i) \in \text{Hom}(A, \mathcal{O}_T(T)) \times \text{Hom}(B, \mathcal{O}_T(T)) : \pi_X^i \circ f^i = \pi_Y^i \circ g^i\}$.

This is now a statement entirely in terms of rings, and is in fact the universal property of the tensor product. However, it is not hard to check directly: given the pair of morphisms $(f^i, g^i)$, we define a morphism $A \otimes_C B \to \mathcal{O}_T(T)$ by $a \otimes b \mapsto f^i(a)g^i(b)$, checking that the composition condition on $f^i$ and $g^i$ means that the map is well-defined. □

We have now verified that the functor $X \times_Z Y$ is a Zariski sheaf, so in principle, it could be representable. Moreover, the functor is representable in the affine case, and since every scheme has an open cover by affines, one might well speculate that it should be possible to glue together affine fibered products to get the fibered product in general. In fact, this is true, and is a special case of a rather general statement about Zariski sheaves. In order to proceed further, we need to go back and consider what it would mean to show locally that a functor is representable.

Thus, let $F : \text{Sch}_S \to \text{Set}$ be a representable functor, and suppose that $F$ is represented by $(X, \eta)$, where $X$ is a scheme over $S$, and $\eta$ an element of $F(X)$. Now suppose that $U$ is an open subscheme of $X$ (i.e., a scheme obtained from $X$ by restricting to an open subset). We then get $\eta|_U \in F(U)$. We want to say that $h_U$ is an “open subfunctor” of $h_X$ under the map induced by $\iota$, which then would mean that $(U, \eta')$ represents a certain open subfunctor of $F$.

How can we formulate this precisely? The first statement is quite simple: for any scheme $T$ over $S$, the morphisms $T \to U$ are a subset of the morphisms $T \to X$. The more subtle idea is as follows: for any morphism $f : T \to X$ over $S$, taking the preimage of $U$ gives us an open subscheme $T_U$ of $T$, and a map $f_U : T_U \to U$ (still over $S$). Moreover, we see that if $T' \to T$ is any morphism such that the composition $T' \to X$ has image contained in $U$, then $T' \to T$ factors uniquely through the inclusion $T_U \to T$. More formally, for any element $f \in h_X(T)$, we have an open subscheme $\iota : T_U \to T$ such that: for all $T''$, and $g : T'' \to T$, if $f \circ g \in h_X(T')$ lies in the subset $h_U(T'') \subseteq h_X(T'' \cap T)$, then $g$ factors (necessarily uniquely) through $T_U$.

We can now formalize this situation.

**Definition 4.2.** Let $F : \text{Sch}_S \to \text{Set}$ be a contravariant functor. A subfunctor $G \subseteq F$ consists of a subset $G(T) \subseteq F(T)$ for each $T \in \text{Obj}(\text{Sch}_S)$, such that $G$ has the induced structure of a functor: that is, for all $T' \to T$, we have $G(T)|_{T'} \subseteq G(T'')$. 5
If \( G \subseteq F \) is a subfunctor, we say \( G \) is open if for every \( T \in \text{Obj}(\text{Sch}_S) \), and every \( \eta \in F(T) \), there exists an open subscheme \( U \) of \( T \) such that \( \eta|_U \in G(U) \), and furthermore, for any \( f : T' \to T \), if \( \eta|_{T'} \in G(T') \subseteq F(T') \), then \( f(T') \subseteq U \).

Note that the definition of openness may be stated equivalently as saying \( \eta|_{T'} \in G(T') \) if and only if \( T' \to T \) factors through \( U \), which we can rephrase as saying that \( (U, \eta|_U) \) represents the functor \( \text{Sch}_T \to \text{Set} \) which is the empty set if \( \eta|_{T'} \not\in G(T') \), and the one-point set if \( \eta|_{T'} \in G(T') \).

In particular \( U \) is unique if it exists.

We can then easily talk about a cover of a functor by open subfunctors:

**Definition 4.3.** A collection \( F_i \to F \) of open subfunctors of \( F \) are said to cover \( F \) if for every \( T \) and \( \eta \in F(T) \), if \( U_i \) is the open subscheme of \( T \) associated to \( \eta \) by \( F_i \), then the \( U_i \) cover \( T \).

We observe that if \( U \subseteq X \) is an open subscheme, then \( h_U \) is an open subfunctor of \( h_X \), and if \( \{U_i\} \) is an open cover of \( X \), then \( h_{U_i} \) is a cover by open subfunctors of \( h_X \).

Now, suppose we have a cover of \( F \) by open subfunctors \( F_i \), and each \( F_i \) is represented by \((U_i, \eta_i)\). We want to glue the \((U_i, \eta_i)\) together to give an \((X, \eta)\) representing \( F \). We will see that in order to do this, we will need that \( F \) is a Zariski sheaf. However, the statement itself is surprisingly simple.

**Theorem 4.4.** Suppose that \( F : \text{Sch}_S \to \text{Set} \) is a Zariski sheaf, and that \( F \) has a cover by open subfunctors \( F_i \subseteq F \). Suppose further that each \( F_i \) is representable. Then \( F \) is representable.

More precisely, if each \( F_i \) is represented by \((U_i, \eta_i)\), with \( U_i \in \text{Obj}(\text{Sch}_S) \) and \( \eta_i \in F_i(U_i) \), then there exists \((X, \eta)\) representing \( F \), with maps \( \psi_i : U_i \to X \) such that:

(i) each \( U_i \) maps isomorphically to an open subscheme of \( X \);

(ii) the \( \psi_i(U_i) \) cover \( X \);

(iii) \( (F(\psi_i))(\eta) = \eta_i \) for each \( i \).

Before we give the proof, we need to understand what is necessary in general to glue together schemes, without worrying about functors. We have:

**Proposition 4.5.** Let \( \{U_i\} \) be a collection of schemes, and for each \( i \neq j \), suppose we have an open subscheme \( U_{i,j} \subseteq U_i \). Suppose we also have for every \( i \neq j \) an isomorphism \( \varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i} \), satisfying:

(i) for each \( i \neq j \), we have \( \varphi_{i,j} = (\varphi_{j,i})^{-1} \);

(ii) for each \( i, j, k \) pairwise distinct, \( \varphi_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k} \), and \( \varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j} \) on \( U_{i,j} \cap U_{i,k} \).

Then we can glue together the \( U_i \) along the \((U_{i,j}, \varphi_{i,j})\) to obtain a scheme \( X \). More precisely, there is a scheme \( X \), and morphisms \( \psi_i : U_i \to X \), such that:

(i) each \( \psi_i \) is an isomorphism onto an open subscheme of \( X \);

(ii) the \( \psi_i(U_i) \) cover \( X \);

(iii) for each \( i \neq j \), we have \( \psi_i(U_{i,j}) = \psi_i(U_i) \cap \psi_j(U_j) \);

(iv) for each \( i \neq j \), we have \( \psi_i = \psi_j \circ \varphi_{i,j} \) on \( U_{i,j} \).

**Proof.** Left as an exercise (Exercise 2.12 of Chapter II of [1]).

Before giving the proof of Theorem 4.4, we observe that we can “intersect” two open subfunctors to obtain a new open subfunctor, in the obvious way: given \( G, G' \) two open subfunctors of \( F \), we consider \( G(T), G'(T) \subseteq F(T) \) under the maps provided, and get a new functor by intersecting inside \( F(T) \) for every \( T \). If \( U, U' \subseteq T \) are the open subsets provided by (ii) of the definition, we check that \( U \cap U' \) works for the intersected functor.

**Proof of Theorem 4.4.** We first use the \( \eta_i \) to glue together the \( U_i \) into a single scheme \( X \). According to the previous proposition, we need open subschemes \( U_{i,j} \subseteq U_i \) and isomorphisms \( \varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i} \).
satisfying the stated compatibility conditions. The idea is simple: by the definition of an open
subfunctor, if we are given \( i \neq j \), then associated to \( \eta_i \in F(U_i) \), the open subfunctor \( F_j \) gives us
an open subscheme, which we denote \( U_{i,j} \), of \( U_i \), and an element \( \eta_{i,j} \in F_j(U_{i,j}) \) such that \( \eta_i \) agrees
with \( \eta_{i,j} \) in \( F(U_{i,j}) \), and such that \( (U_{i,j}, \eta_{i,j}) \) represents the functor of morphisms \( T' \to U_i \) such that
\( \eta_{i,T'} \in F_j(T') \subseteq F(T) \).

One checks that in fact \( (U_{i,j}, \eta_{i,j}) \) represents \( F_i \cap F_j \). It follows that \( (U_{i,j}, \eta_{i,j}) \) and \( (U_{j,i}, \eta_{j,i}) \)
represent the same functor, so by Yoneda’s lemma we obtain a unique isomorphism \( \varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i} \)
sending \( \eta_{j,i} \) to \( \eta_{i,j} \). We then claim that the \( U_{i,j} \) and \( \varphi_{i,j} \) satisfy the conditions of Proposition 4.5.
Indeed, the first condition is immediate, while the second follows from writing everything in terms
of the functor \( F_i \cap F_j \cap F_k \), and following through the definitions, again using Yoneda’s lemma.

It thus follows that we obtain a scheme \( X \) and maps \( \psi_i : U_i \to X \) as in the proposition. It is then
easy to show that we have an \( \eta \in F(X) \) such that \( (X, \eta) \) represents \( F \), and \( \eta|_{U_i} = \eta_i \) for each \( i \).
Here we finally use that \( F \) is a Zariski sheaf: for each \( i \), we can consider \( \eta_i \in F_i(U_i) \) as an element
of \( F(U_i) \), and considering the \( U_i \) as open subsets of \( X \), following through the definitions we find
that \( \eta_{i,U_i \cap U_j} = \eta_j|_{U_i \cap U_j} \) for every \( i, j \). Thus, by the Zariski sheaf condition, the \( \eta \) glue uniquely
to give an element \( \eta \in F(X) \) with \( \eta|_{U_i} = \eta_i \) for all \( i \). We claim that \( (X, \eta) \) represents \( F \). Indeed,
given \( T \) and \( \zeta \in F(T) \), the \( F_i \) give us an open cover \( T_{U_i} \) of \( T \), and because \( (U_i, \eta_i) \) represent \( F_i \),
we get morphisms \( T_{U_i} \to U_i \), with \( \zeta_{T_{U_i}} = \eta_i|_{T_{U_i}} \), and which agree on the intersections. We can
dependence to obtain a morphism \( T \to X \), with \( \zeta = \eta|_T \), and it follows by definition that \( (X, \eta) \)
represent \( F \).

The final proposition we will need to prove existence of fibered products is the following.

**Proposition 4.6.** Given \( \pi_X : X \to Z \), \( \pi_Y : Y \to Z \), let \( U, V, W \) be open subschemes of \( X, Y, Z \)
respectively such that \( U \subseteq \pi_X^{-1}(W) \), \( V \subseteq \pi_Y^{-1}(W) \). Then \( U \times_W V \) is naturally an open subfunctor
of \( X \times_Z Y \).

Furthermore, if \( \{(U_i, V_i, W_i)\} \) are a family such that for any \( P \in X, Q \in Y \) with the same image
in \( Z \), there is some \( i \) with \( P \in U_i \) and \( Q \in V_i \), the subfunctors \( U \times_W V \) cover \( X \times_Z Y \).

**Proof.** This is really just a formality. We certainly have a morphism \( U \times_W V \to X \times_Z Y \) by
composing with the inclusions \( U \to X, V \to Y \). Moreover, the map \( U \times_W V(T) \to X \times_Z Y(T) \)
is clearly injective for any \( T \). Now, given morphisms \( f : T \to X, g : T \to Y \) agreeing on \( Z \), the
open subscheme \( f^{-1}(U) \cap g^{-1}(V) \subseteq T \) clearly has the property that for any \( T' \to T \) with images
contained in \( U \subseteq X \) and \( V \subseteq Y \), we must have \( T' \to T \) factoring uniquely through \( f^{-1}(U) \cap g^{-1}(V) \).
Thus, \( U \times_W V \) is an open subfunctor of \( X \times_Z Y \).

Finally, given a collection \( \{(U_i, V_i, W_i)\} \), and a pair of morphisms \( f : T \to X, g : T \to Y \)
agreeing on \( Z \), let \( t \in T \) be any point, with images \( P \in X \) and \( Q \in Y \); then \( P \) and \( Q \) have the
same image in \( Z \). By assumption, there is an \( i \) such that \( P \in U_i \), and \( Q \in V_i \), so it follows that
\( t \in f^{-1}(U_i) \cap g^{-1}(V_i) \). Thus, our open subsets cover \( T \), and our open subfunctors cover \( X \times_Z Y \),
as desired.

We can now easily prove the existence of fibered products.

**Theorem 4.7.** Fibered products exist in the category of schemes, and therefore also in the category
of schemes over \( S \) for any fixed scheme \( S \).

**Proof.** From the statement of the universal property, we see that if \( X \times_Z Y \) is a fibered product
in the category of all schemes, and \( X, Y, Z \) happen to be schemes over \( S \), then \( X \times_Z Y \) is also a
fibered product in the category of schemes over \( S \). Thus, it is enough to work in the full category
of schemes.

By Proposition 3.4, the functor \( X \times_Z Y \) is a Zariski sheaf. By Theorem 4.4, it therefore suffices
to produce a cover by open subfunctors which are each representable. By Propositions 4.6 and 4.1,
we are done if we can produce a family of affine open subsets \( \{(U_i, V_i, W_i)\} \) of \( X, Y, Z \) respectively such that: \( U_i \) and \( V_i \) map into \( W_i \subseteq Z \) for all \( i \); and, for any \( P \in X, Q \in Y \) with the same image in \( Z \), there is some \( i \) with \( P \in U_i \) and \( Q \in V_i \). But if we simply let \( \{(U_i, V_i, W_i)\} \) be the family of all such affine open subsets, we see this has the desired property, as there is some \( W \) open affine containing \( \pi_X(P) = \pi_Y(Q) \), and then some affine open \( U \) in \( \pi_X^{-1}(W) \) containing \( P \), and some affine open \( V \) in \( \pi_Y^{-1}(W) \) containing \( Q \). Thus, we conclude that \( X \times_Z Y \) is representable, as desired. \( \square \)

While this may seem like a lot of machinery to prove that fibered products exist, the advantage is that we have done most of our work as part of a more general framework, which will also apply, for instance, to proving that Grassmannians exist. Hopefully, it also illuminates more clearly the concepts behind the standard proof.

**Remark 4.8.** The proof that fibered products exist use open covers coming from reductions to the affine case, but in general other open covers also arise naturally. For instance, in order to prove that \( G(r, d) \) is representable, we consider open subfunctors consisting of \( r \)-dimensional subspaces which have trivial intersection with a fixed \((d - r)\)-dimensional subspace. One shows that these open subfunctors are representable, and in fact represented by \( \mathbb{A}^r(d-r) \), and existence of \( G(r, d) \) follows.

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5. THE FIBERED PRODUCT: COMMENTS AND EXAMPLES

We discuss now the fibered product: how to think about it, some examples, and some applications.

Roughly speaking, one should think of the fibered product \( X \times_Z Y \) as parametrizing \((x, y) \in X \times Y \) which map to the same thing in \( Z \). In fact, this is (by definition) precisely correct on the level of \( k \)-valued points, or \( T \)-valued points for any \( T \), but this does not mean that the fibered product is simple to understand.

We first consider the case of very simple varieties over a field \( k \).

**Example 5.1.** Perhaps the most basic example of a fibered product is \( \mathbb{A}^1_k \times_{\text{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k \). Indeed, this follows immediately from the isomorphism \( k[x] \otimes_k k[y] \cong k[x, y] \). We see however that if we look at the underlying topological spaces, the points of the product are not the product of the points, and the topology on the product is far finer than the product topology.

This is true even if \( k \) is algebraically closed. In this case, the points of \( \mathbb{A}^2_k \) aren’t so far from the product of the points of \( \mathbb{A}^1_k \); indeed, the closed points of \( \mathbb{A}^2_k \) are in bijection with pairs of closed points of \( \mathbb{A}^1_k \). The problem comes from non-closed points. If \( \eta \) is the generic point of \( \mathbb{A}^1_k \), the product of the points of \( \mathbb{A}^1_k \) consists of pairs \((x, y)\) where each of \( x \) and \( y \) is either a closed point of \( \mathbb{A}^1_k \) (i.e., an element of \( k \)) or \( \eta \). On the other hand, the points of \( \mathbb{A}^2_k \) also include generic points corresponding to every irreducible curve in the plane (i.e., every irreducible polynomial in \( x, y \)), as well as a generic point for the whole plane (corresponding to the \( 0 \) ideal). We can think of the points \((x_0, \eta)\) for \( x_0 \in k \) as corresponding to the line \( x = x_0 \) in \( \mathbb{A}^2_k \), and \((\eta, y_0)\) corresponds to \( y = y_0 \). Furthermore, it is reasonable for \((\eta, \eta)\) to correspond to the generic point of \( \mathbb{A}^2_k \). However, this means that for any irreducible curve \( C \subseteq \mathbb{A}^2_k \) which is not a vertical or horizontal line, there is a point of \( \mathbb{A}^2_k \) not corresponding to any pair of points of \( \mathbb{A}^1_k \).

One way to think about what is going on is that in the case of varieties, we had that the underlying set of a product was the product set, but the Zariski topology on the product was finer than the product topology. But because in scheme land every irreducible closed subset has a unique generic point, this difference in topologies is now also being reflected in the underlying sets.

We do have the following compatibility:
Proposition 5.2. If $k$ is an algebraically closed field, $X, Y$ prevarieties over $k$, and $X', Y'$ the associated $k$-schemes, then $X' \times_{\text{Spec} \, k} Y'$ is the $k$-scheme associated to $X \times Y$.

The proof is omitted.

Non-algebraically closed fields lead to stranger phenomena.

Example 5.3. Consider $\text{Spec} \, \mathbb{Q}(i) \otimes_{\text{Spec} \, \mathbb{Q}} \text{Spec} \, \mathbb{Q}(i)$. This is the product of a point with a point over a point, and yet $\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i) \cong \mathbb{Q}(i) \times \mathbb{Q}(i)$, which means that we have

$$\text{Spec} \, \mathbb{Q}(i) \times_{\text{Spec} \, \mathbb{Q}} \text{Spec} \, \mathbb{Q}(i) \cong \text{Spec} \, \mathbb{Q}(i) \amalg \text{Spec} \, \mathbb{Q}(i),$$

i.e., the fibered product is the disjoint union of two copies of $\text{Spec} \, \mathbb{Q}(i)$.

On the level of $\mathbb{Q}(i)$-valued points, we note that $\text{Spec} \, \mathbb{Q}(i)$ actually has not one but two $\mathbb{Q}(i)$-valued points, corresponding to the identity map and complex conjugation. Thus, there are four possibilities for pairs of $\mathbb{Q}(i)$-valued points of $\text{Spec} \, \mathbb{Q}(i)$, and $\text{Spec} \, \mathbb{Q}(i) \amalg \text{Spec} \, \mathbb{Q}(i)$ likewise has four $\mathbb{Q}(i)$-valued points, as it must.

Fibered products of schemes can also lend some geometric intuition to an odd algebraic fact.

Example 5.4. A common example of the strangeness of tensor products is the fact that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ is the zero ring. However, we now have a very geometric way of thinking about this fact: $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ being zero is equivalent to $\text{Spec}(\mathbb{Z}/2\mathbb{Z}) \times_{\text{Spec} \, \mathbb{Z}} \text{Spec}(\mathbb{Z}/3\mathbb{Z})$ being empty.

But for any $p$, $\text{Spec}(\mathbb{Z}/p\mathbb{Z})$ is a point lying over the point of $\mathbb{Z}$ corresponding to the prime ideal $(p)$, and it is clear that if we have $X \to Z$ and $Y \to Z$ with disjoint images, then $X \times_Z Y$ can’t have any points, and must be empty!

We now discuss some applications of the fibered product.

Recall that if $f : X \to Y$ is a continuous map of topological spaces, and $y \in Y$, the fiber of $f$ over $y$ is the set $f^{-1}(y)$, equipped with the topology inherited from $X$. The fibered product allows us to define a scheme-theoretic version of fibers, as follows:

Definition 5.5. Suppose $f : X \to Y$ is a morphism of schemes, and $y \in Y$ a point. The fiber $X_y$ of $f$ over $y$ is defined to be $\text{Spec} \, \kappa(y) \times_Y X$, where $\text{Spec} \, \kappa(y) \to Y$ is the unique morphism with image $y$.

This immediately gives us a scheme structure, and we have:

Exercise 5.6. The projection morphism $\text{Spec} \, \kappa(y) \times_Y X \to X$ is a homeomorphism onto the topological fiber $f^{-1}(y)$.

This gives rise to the point of view of a morphism as a family of schemes: if we have a morphism $f : X \to Y$, then for every point $y \in Y$ we have the fiber $X_y$, so we can think of $f$ as giving a family of schemes parametrized by (the points of) $Y$.

More generally, we have the following:

Definition 5.7. If $f : X \to Y$ is a morphism, and $Y' \to Y$ is any other morphism, the natural map $X \times_Y Y' \to Y'$ is called the base change (base extension, in $[1]$) of $f$ under $Y' \to Y$.

This is an important concept in general, but we mention one special case which has a simple classical meaning: if $Y = \text{Spec} \, k$, and $Y' = \text{Spec} \, k'$ for some $k' \supseteq k$, then the base change $X' := X \times_{\text{Spec} \, k} \text{Spec} \, k' \to \text{Spec} \, k'$ gives us a scheme over $\text{Spec} \, k'$ starting from one over $\text{Spec} \, k$. If $X$ were an affine variety, given by polynomials with coefficients in $k$, then $X'$ is simply the variety we get by considering the polynomials as having coefficients in the larger field $k'$. In particular, if $k' = k$, then $X'$ is a variety over an algebraically closed field, and it is frequently useful to take what we know in this case and see what we can conclude about our original $X$. This sort of extension of fields is classical and concrete, but still a bit subtle in general: note that Example 5.3 can be viewed as a special case.
REFERENCES