Exercise 1. Given $X \subseteq \mathbb{A}^n_k$ an affine algebraic set, and $P \in X$, we can naturally imbed $T_P(X) \subseteq k^n$ as an affine-linear subspace of $\mathbb{A}^n_k$ by sending $v \in k^n$ to $P + v \in \mathbb{A}^n_k$. Show that if $X$ is defined by homogeneous polynomials, then this imbedding of $T_P(X)$ in $\mathbb{A}^n_k$ is likewise defined by homogeneous polynomials (equivalently, contains the origin).

Now, given $X \subseteq \mathbb{P}^n_k$ a projective algebraic set, and $P \in X$, we can think of $T_P(X)$ as yielding a linear subvariety of $\mathbb{P}^{n+1}_k$, which we will denote by $\overline{T}_P(X)$, as follows: let $Y \subseteq \mathbb{A}^{n+1}_k$ be the affine cone over $X$, and choose $Q \in Y$ mapping to $P$. Using Exercise 1, imbed $T_Q(Y)$ in $\mathbb{A}^{n+1}_k$, and let $\overline{T}_P(X)$ be the projective algebraic set over which $T_Q(Y)$ is the affine cone. Thus, $\overline{T}_P(X)$ is a linear subspace of $\mathbb{P}^n_k$ of dimension equal to the dimension of $T_P(X)$. But while $T_P(X)$ is a vector space, $\overline{T}_P(X)$ is a projective linear space.

Exercise 2. If $X \subseteq \mathbb{P}^n_k$ is a projective algebraic set, we can define the tangent variety of $X$ to be the set
\[ \text{Tan}(X) = \{ P \in \mathbb{P}^n_k : \exists Q \in X \text{ such that } P \in \overline{T}_Q(X) \}. \]
Show that Tan$(X)$ is an algebraic set, and that if $X$ is irreducible and nonsingular of dimension $d$, then $\dim \text{Tan}(X) \leq 2d$.

Exercise 3. Conclude that in Proposition 9.5.1 of the lecture notes, we could have required that $\varphi$ is in fact an isomorphism onto a subvariety of $\mathbb{P}^3_k$. 

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