Definition. Given a field $k$, a CSA over $k$ is a $k$-algebra $R$ which is finite-dimensional over $k$, has center equal to $k$, and has no two-sided ideals other than 0 and $R$.

CSA stands for “central simple algebra”, which is the standard terminology, but does not mesh well with Lang’s usage.

Exercise 1. Show that a $k$-algebra is a CSA over $k$ if and only if it is isomorphic to $\text{Mat}_n(D)$ for some $n \geq 1$ and some division ring $D$ which is itself a CSA over $k$.

Exercise 2. If $R, S$ are CSAs over $k$, then $R \otimes_k S$ has no two-sided ideals other than 0 and $R \otimes_k S$.

Hint: let $I$ be a nonzero two-sided ideal in the tensor product, and let $x \in I$ be a nonzero element chosen so among all nonzero elements of $I$, if we write $x = \sum_i r_i \otimes s_i$ with the $s_i$ linearly independent over $k$, we obtain the minimal possible number of terms in the sum. Show that if we replace $x$ by an element with the same number of simple tensors in its sum, we can assume one of these simple tensors is of the form $1 \otimes s_i$. Show then that $(r \otimes 1)x - x(r \otimes 1) = 0$, and use this to produce an element in $I$ of the form $1 \otimes s_i$, with $s \in S$ nonzero. Conclude that $I = R \otimes_k S$.

Note: you will not need to use that the center of $S$ is equal to $k$ in the above exercise (nor in fact that either $R$ or $S$ is finite-dimensional).

Exercise 3. Give an example of a field $k$, and $k$-algebras $R$ and $S$ which are finite-dimensional over $k$, and have no nontrivial two-sided ideals, but such that $R \otimes_k S$ has nontrivial two-sided ideals.

Hint: this is easier than you probably think it is.

Exercise 4. If $R, S$ are CSAs over $k$, then $R \otimes_k S$ is a CSA over $k$.

Hint: represent an element in the center as $\sum_i r_i \otimes s_i$ with the $s_i$ linearly independent, and use the fact that in particular this element must commute with $R \otimes 1$ and $1 \otimes S$.

Exercise 5. If $R$ is a CSA over $k$, then $R \otimes_k R^o$ is isomorphic to a matrix algebra over $k$.

Hint: construct a $k$-algebra homomorphism to $\text{End}_k(R)$ via left and right scalar multiplication maps, show that it is injective, and use the dimensions to conclude it is an isomorphism.

The significance of these exercises is as follows: if we take the set of isomorphism classes of CSAs over a given field $k$, we have an operation given by tensor product, which makes the set into a commutative semigroup. If we mod out by the equivalence relation generated by $R \sim \text{Mat}_n(R)$, we actually obtain a group, called the Brauer group of $k$, with the inverse of $R$ given by $R^o$. We have shown that the Brauer group of an algebraically closed field is trivial, but the Brauer group of other fields is an important topic in algebraic number theory and algebraic geometry. Part of the reason for this is that the Brauer group has an interpretation as the Galois cohomology group $H^2(\text{Gal}(\bar{k}/k), \bar{k}^*)$ (here if $k$ is not perfect one should technically use the separable closure rather than the algebraic closure).