Exercise 1. Suppose $F$ is a field of characteristic $\neq 2$. An extension $K/F$ is **biquadratic** if $K = F(\sqrt{m}, \sqrt{n})$ for some $m, n \in F$, and $[K : F] = 4$. Show that if $m, n$ are not perfect squares in $F$, then $F(\sqrt{m}, \sqrt{n})$ is biquadratic if and only if $mn$ is not a perfect square in $F$. Show further that if $F(\sqrt{m}, \sqrt{n})$ is biquadratic, then $F(\sqrt{m}, \sqrt{n}) = F(\sqrt{m} + \sqrt{n})$.

Exercise 2. Suppose $F$ is a field of characteristic $\neq 2$, and $a, b \in F$ with $b$ not a square. Show that $\sqrt{a} + \sqrt{b} = \sqrt{m} + \sqrt{n}$ for some $m, n \in F$ if and only if $a^2 - b$ is a square in $F$.

When is $F(\sqrt{a} + \sqrt{b})$ a biquadratic extension of $F$?

Exercise 3. A field $F$ is **formally real** if $-1$ cannot be written as a sum of (any number of) squares in $F$. Suppose $F$ is formally real, and $f(x) \in F[x]$ an irreducible polynomial of odd degree. Letting $\alpha$ be a root of $f(x)$ in some extension of $F$, show that $F(\alpha)$ is still formally real. (Hint: consider a counterexample of minimal degree, and then use the definitions to produce a counterexample of smaller degree)

Exercise 4. Let $K/F$ be an algebraic field extension, and suppose $R$ is any subring of $K$ containing $F$. Show that $R$ is in fact a field.

Exercise 5. Suppose we have extensions $K/F$ and $E/F$, both contained in some larger field.

(a) Show that if $K/F$ is normal, then $EK/E$ is normal.

(b) Show that if $K/F$ and $E/F$ are both normal, then $KE/F$ is normal.