1. Conclusion of proof

We had written

\[ \zeta_{K,C}(s) = N(I_0)^s \sum_{\bar{z} \in \varphi(I_0) \cap X_{K,a}} \frac{1}{N(\bar{z})^s}. \]

To complete the proof of the analytic class number formula, we now finish the proof of the following general theorem:

**Theorem 1.1.** Let \( X \subset \mathbb{R}^n \) be any cone containing only points \( \bar{z} \) with \( N(\bar{z}) \neq 0 \), and such that \( X \cap \{ \bar{z} \in \mathbb{R}^n : N(\bar{z}) \leq 1 \} \) is bounded and has some volume \( v_X \). Let \( L \subset \mathbb{R}^n \) be a lattice of full rank with volume \( v_L \). Define

\[ \zeta_{X,L}(s) = \sum_{\bar{z} \in X \cap L} \frac{1}{N(\bar{z})^s}. \]

Then this sum converges for \( s > 1 \), and we have

\[ \lim_{s \to 1^+} (s - 1) \zeta_{X,L}(s) = \frac{v_X}{v_L}. \]

**Proof.** We had arranged the points of \( L \cap X \) by non-decreasing norm, so that we have \( L \cap X = \{ x_1, x_2, \ldots \} \) with \( N(x_i) \leq N(x_{i+1}) \) for all \( i \). We showed that

\[ \lim_{k \to \infty} \frac{k}{N(x_k)} = \frac{v_X}{v_L}. \]

We now take as well-known that \( \zeta_Q(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) converges for \( s > 1 \) (this follows trivially from the integral test for convergence). We then have \( \zeta_{X,L}(s) = \sum_{k=1}^{\infty} \frac{1}{N(x_k)^s} \), so it follows that \( \zeta_{X,L}(s) \) converges for \( s > 1 \), since

\[ \lim_{k \to \infty} k^s \frac{1}{N(x_k)^s} = \left( \frac{v_X}{v_L} \right)^s, \]

which is non-zero.

For any \( \epsilon > 0 \), from the same equation we also see that for \( k_0 \) sufficiently large, for all \( k \geq k_0 \) we have

\[ \left( \frac{v_X}{v_L} - \epsilon \right) \frac{1}{k} < \frac{1}{N(x_k)} < \left( \frac{v_X}{v_L} + \epsilon \right) \frac{1}{k}. \]

Thus, for \( s > 1 \) we have

\[ \left( \frac{v_X}{v_L} - \epsilon \right)^s \sum_{k=k_0}^{\infty} \frac{1}{k^s} < \sum_{k=k_0}^{\infty} \frac{1}{N(x_k)^s} < \left( \frac{v_X}{v_L} + \epsilon \right)^s \sum_{k=k_0}^{\infty} \frac{1}{k^s}. \]
Multiplying by \((s - 1)\) and letting \(s\) go to 1 from above, we note that the middle sum goes to 
\[
\lim_{s \to 1^+} (s - 1) \zeta_{X,L}(s),
\]
and similarly, the sums on the left and right go to 
\[
\lim_{s \to 1^+} (s - 1) \zeta_{Q}(s).\]
We thus obtain
\[
\left(\frac{v_X}{v_L} - \epsilon\right) \lim_{s \to 1^+} (s - 1) \zeta_{Q}(s) \leq \lim \inf_{s \to 1^+} (s - 1) \zeta_{X,L}(s)
\]
\[
\leq \lim \sup_{s \to 1^+} (s - 1) \zeta_{X,L}(s) \leq \left(\frac{v_X}{v_L} + \epsilon\right) \lim_{s \to 1^+} (s - 1) \zeta_{Q}(s),
\]
so if we show that 
\[
\lim_{s \to 1^+} (s - 1) \zeta_{Q}(s) = 1,
\]
we conclude the desired statement for \(\zeta_{X,L}(s)\).

But this also follows from a slightly refined version of the integral test: we note that 
\[
1 = \int_1^{\infty} \frac{dx}{x^s} < \zeta_{Q}(s) < 1 + \int_1^{\infty} \frac{dx}{x^s} = 1 + \frac{1}{s - 1},
\]
and multiplying by \((s - 1)\) and taking the limit as \(s\) goes to 1 from above gives the desired statement. \(\Box\)

Remark 1.2. In fact, this proof does not use in an essential way the definition of the function \(N(\bar{z})\), and goes through if \(N(\bar{z})\) is replaced by any function \(F: X \to \mathbb{R}_{>0}\) such that \(F(t\bar{z}) = t^n F(\bar{z})\) for any \(\bar{z} \in X\) and \(t \in \mathbb{R}_{>0}\), and such that \(X \cap \{\bar{z}: F(\bar{z}) \leq 1\}\) is bounded with volume \(v_{X,F}\); in the formula of the theorem, we replace \(v_{X,L}\) by \(v_{X,F}\).

Recalling that in Theorem 1.2 of lecture 7 we already calculated
\[
\text{vol } \varphi(I_0) = \frac{N(I_0) |D_K|^{1/2}}{2^{r_2}},
\]
the previous two theorems together complete the proof of the analytic class number formula, which we restate for convenience:

**Theorem 1.3.** Let \(K\) be a number field. Then:

(i) The sum for \(\zeta_K(s)\) converges for \(\Re s > 1\).
(ii) (Analytic class number formula) \(\zeta_K(s)\) has a simple pole at \(s = 1\), with residue given by
\[
\lim_{s \to 1^+} (s - 1) \zeta_K(s) = \frac{2^{r_1 + r_2} \pi^{r_2} h_K R_K}{m_K |D_K|^{1/2}}.
\]
(iii) (Euler product) For \(s > 1\), we can also write
\[
\zeta_K(s) = \prod_p \left(1 - \frac{1}{N(p)}\right),
\]
where \(p\) ranges over non-zero prime ideals of \(\mathcal{O}_K\).

Remark 1.4. We will discuss several applications of the analytic class number formula, but we also remark that it is the template for the famous conjecture of Birch and Swinnerton-Dyer, when one replaces the zeta function of a number field by the \(L\)-function of an elliptic curve.

We will now begin a study of the case of subfields of cyclotomic fields, with the goal of concrete applications of the analytic class number formula to both the cyclotomic and quadratic case (and some elementary consequences of each). The
first step will be to use the Euler product to give a different description of the zeta function in this situation. We will need to develop some background on Dirichlet characters and $L$-functions.

2. Dirichlet characters

**Definition 2.1.** A **Dirichlet character** is a multiplicative homomorphism $\chi : \mathbb{Z}/f\mathbb{Z}^* \to \mathbb{C}^*$, such that for any $d | f$, $\chi$ does not induce a map $\mathbb{Z}/d\mathbb{Z}^* \to \mathbb{C}^*$; then $f$ is called the **conductor** of $\chi$. We will also consider $\chi$ as a multiplicative map $\mathbb{Z} \to \mathbb{C}$ by defining

$$\chi(x) = \begin{cases} \chi(\bar{x}) : (x, f) = 1 \\ 0 : \text{otherwise} \end{cases}.$$ 

**Remark 2.2.** In fact, these characters are usually called “primitive”, and our convention of always working with primitive characters differs some other sources, which prefer to work with groups of characters of a fixed modulus. However, for our purposes, the formulas will be simpler with this convention. It does, however, mean that we will have to make a slightly more complicated convention of multiplication of characters.

Although $\mathbb{Z}/1\mathbb{Z}^*$ is not a very good concept, we set the following convention:

**Notation 2.3.** We denote by $\chi_1$ the trivial character, which is the unique character considered to be of conductor 1, and corresponds to the constant map $\mathbb{Z} \to \mathbb{C}$ with value 1.

Given a Dirichlet character $\chi$, we also denote by $\bar{\chi}$ the complex conjugate character defined by $\bar{\chi}(a) = \chi(a)$.

**Definition 2.4.** We say that $\chi$ is a **Dirichlet character modulo** $n$ if it is a Dirichlet character of conductor $f$ with $f|n$; we then have that $\chi$ induces a multiplicative map $\mathbb{Z}/n\mathbb{Z}^* \to \mathbb{C}^*$. Conversely, given a multiplicative map $\mathbb{Z}/n\mathbb{Z}^* \to \mathbb{C}^*$, we obtain a unique Dirichlet character (which will be modulo $n$) as $\chi : \mathbb{Z}/f\mathbb{Z}^* \to \mathbb{C}^*$, where $f|n$ is the smallest integer such that the original map remains defined modulo $f$.

**Definition 2.5.** Given characters $\chi$ and $\psi$ of conductors $f_\chi$ and $f_\psi$, we obtain maps $\chi, \psi : \mathbb{Z}/(\text{lcm}(f_\chi, f_\psi))\mathbb{Z}^* \to \mathbb{C}^*$, so by taking products we obtain a map $\mathbb{Z}/(\text{lcm}(f_\chi, f_\psi))\mathbb{Z}^* \to \mathbb{C}^*$. We thus let the **product character** $\chi \psi$ be the Dirichlet character associated to this map.

**Easy Facts 2.6.** We make the following observations about Dirichlet characters:

(i) If $\chi, \psi$ are Dirichlet characters modulo $n$, then $\chi \psi$ is a Dirichlet character modulo $n$.

(ii) For any Dirichlet character $\chi$, we have $\chi \bar{\chi} = \chi_1$.

(iii) The set of Dirichlet characters modulo $n$ form a group under multiplication.

From our part of view, the significance of Dirichlet characters is that they may also be viewed as characters on the Galois group of cyclotomic fields:

**Lemma 2.7.** $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \mathbb{Z}/n\mathbb{Z}^*$ (i.e., there is a canonical isomorphism). Thus, every Dirichlet character modulo $n$ may be viewed as a homomorphism $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \to \mathbb{C}^*$.
Proof. Although the group of $n$th roots of unity is non-canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and in particular, the choice of $\zeta_n$ is non-canonical, for any choice of $\zeta_n$, we see that an automorphism of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is uniquely determined by $\zeta_n^a \mapsto (\zeta_n^a)^a$, we see that $a$ is independent of the choice of $\zeta_n$. □

Therefore, we can make the following definition:

Definition 2.8. Let $G$ be a subgroup of the group of Dirichlet characters modulo $n$. We define the associated field $K_G \subset \mathbb{Q}(\zeta_n)$ to be the fixed field of $G' \subset \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, where $G'$ is the subgroup of elements in the kernel of every $\chi \in G$.

We can also go in the other direction:

Definition 2.9. Given $K \subset \mathbb{Q}(\zeta_n)$, we can define a group $G_K$ of Dirichlet characters modulo $n$ by taking all characters $\chi : \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \to \mathbb{C}^*$ which are trivial when restricted to $\text{Gal}(\mathbb{Q}(\zeta_n)/K) \subset \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$.

Basic theory of characters of finite abelian groups, together with basic Galois theory, give:

Proposition 2.10. The maps $G \mapsto K_G$ and $K \mapsto G_K$ are inverse to one another, giving a natural (inclusion-preserving) bijection between groups of Dirichlet characters modulo $n$ and subfields of $\mathbb{Q}(\zeta_n)$.

For the purpose of examples, it will be useful to consider the following:

Definition 2.11. We say that a Dirichlet character $\chi$ is odd if $\chi(-1) = -1$; otherwise, $\chi(-1) = 1$, and we say $\chi$ is even.

We conclude by discussing three important special cases, which we will return to later:

Example 2.12. If $G$ is the full group of Dirichlet characters modulo $n$, one easily checks that $G'$ is trivial, so that $K_G = \mathbb{Q}(\zeta_n)/\mathbb{Q}$.

If $G$ is the subgroup of even Dirichlet characters modulo $n$, then $G'$ consists of the identity and complex conjugation, and $K_G = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

Finally, if $n$ is odd and square-free, and $G$ consists only of $\chi_1$ and $\chi_n$, which we set to be the Jacobi symbol modulo $n$, we have that $G'$ has index 2 in the group of Dirichlet characters modulo $n$, and $K_G = \mathbb{Q}(\sqrt{\chi_n(-1)n})$. In particular, $K_G$ is real if and only if $G$ consists of even characters.