The following exercise sounds complicated, but is a situation one frequently runs into in making arguments involving degenerations.

**Exercise 1.** Suppose we are given \( f : X \rightarrow Y \), with \( X \) and \( Y \) Noetherian and irreducible, \( f \) open and of finite type, and \( X \) regular and \( Y \) universally catenary. Suppose further that every component of every fiber of \( f \) has some given dimension \( d \). Given \( Z_1, \ldots, Z_n \) closed subschemes of \( X \) of codimensions \( d_1, \ldots, d_n \), set \( \rho = d - \sum_i d_i \), and suppose \( \rho \geq 0 \).

Set \( Z = \bigcap_i Z_i \subseteq X \). Suppose we are given a point \( z \in Z \), with image \( f(z) = y \) in \( Y \). Show that every component of \( f^{-1}(y) \cap Z \) has dimension at least \( \rho \). Suppose that \( f^{-1}(y) \cap Z \) has dimension exactly \( \rho \). Show that every neighborhood \( U \) of \( z \) in \( Z \) maps dominantly to \( Y \).

We give Nagata’s example of an infinite-dimensional Noetherian ring.

**Exercise 2.** (a) Suppose that \( R \) is a ring such that: (i) for every maximal ideal \( m \), we have \( R_m \) Noetherian; and (ii) every non-zero element \( r \in R \) is contained in only finitely many maximal ideals of \( R \). Show that \( R \) is Noetherian.

(b) Now let \( R_0 = k[x_1, x_2, \ldots] \) be the polynomial ring in infinitely many variables over a field \( k \), and choose an increasing function \( d : \mathbb{N} \rightarrow \mathbb{N} \). Define a sequence of prime ideals \( P_i \) of \( R_0 \) by \( P_i = (x_{d(i)}^1, \ldots, x_{d(i+1)}) \), and set \( S \) to be the multiplicative set \( R_0 \setminus \bigcup_{i=1}^\infty P_i \). Finally, let \( R = S^{-1}R_0 \). Show that the maximal ideals of \( R \) are the ideal \( S^{-1}P_i \).

(c) Conclude that for an appropriate choice of \( d \), the \( R \) of (b) is a Noetherian ring of infinite dimension.

The following exercise is intended to provide some practice with regular and local complete intersection rings, and cotangent spaces.

**Exercise 3.** (a) Let \( \varphi : A \rightarrow B \) be a surjection of Noetherian local rings, inducing an isomorphism on cotangent spaces, and suppose further that \( B \) is regular. Show that \( \varphi \) is an isomorphism. [Geometrically, this says that a regular scheme is as large as possible for its cotangent space.] Hint: you may use that a regular local ring is necessarily an integral domain.

(b) Give an example to show that (a) fails if we assume instead that \( A \) is regular.

(c) Use (a) to show that if \( X \subseteq Y \) is a closed subscheme, with both \( X \) and \( Y \) regular and irreducible, then \( X \) is necessary a local complete intersection in \( Y \).

**Exercise 4.** Do Hartshorne, Exercise 9.1 of Chapter III.

**Exercise 5.** Do Hartshorne, Exercise 9.3 of Chapter III.