Exercise 1. (Gluing morphisms of schemes) Prove the following:

Let $X$ and $Y$ be schemes over $S$, and $\{U_i\}$ an open covering of $X$. Then morphisms $f : X \to Y$ over $S$ are in one-to-one correspondence with collections of morphisms $f_i : U_i \to Y$ over $S$, such that for all $i, j$ we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as morphisms $U_i \cap U_j \to Y$ over $S$.

Exercise 2. (Gluing schemes) Do Hartshorne, Exercise 2.12 of Chapter II.

Exercise 3. Let $S$ be a scheme, and $\mathcal{C}$ a half-full subcategory of $\text{Sch}_S$, such that if $T$ is in $\mathcal{C}$, then every open subscheme of $T$ is in $\mathcal{C}$. Given $X \in \text{Sch}_S$, let $F : \mathcal{C} \to \text{Set}$ be the functor $T \mapsto \text{Mor}_{\text{Sch}_S}(T, X)$ (that is, if $X'$ represents $F$, then $X'$ is universal for maps from $\mathcal{C}$ to $X$).

Show that $F$ is a Zariski sheaf.

A scheme $X$ is integral if for every $U$, we have $\mathcal{O}_X(U)$ is an integral domain. This is equivalent to $X$ being irreducible and reduced; see p. 82 of Hartshorne. A morphism $f : X \to Y$ is dominant if $f(X)$ is dense in $Y$. If $X$ is integral, we say that $X$ is normal if for every $P \in X$, we have that the stalk $\mathcal{O}_{X, P}$ is (in addition to being an integral domain) integrally closed in its field of fractions.

Exercise 4. In the above exercise, suppose that $X$ is integral, and let $\mathcal{C}$ be the category of normal schemes over $S$, with morphisms consisting only of dominant morphisms. Show that the $F$ of the previous exercise is representable by first handling the case that $X$ is affine, and then showing that considering an open affine cover of $X$ gives a cover of $F$ by open subfunctors.

This implies that the normalization of an integral scheme exists: i.e., there is a scheme $X' \to X$ such that $X'$ is normal, and any dominant morphism from a normal scheme to $X$ factors uniquely through $X'$.

Exercise 5. Let $C_1$ be the plane curve given by $y^2 = x^3 + x^2$, and $C_2$ by $y^2 = x^3$. What do the real points of $C_1$ and $C_2$ look like? What are the normalizations of $C_1$ and $C_2$?

Exercise 6. Do Hartshorne, Exercise 3.10 of Chapter II.

Exercise 7. Do Hartshorne, Exercise 3.15 of Chapter II.

The following is a warm-up for proving that the Grassmannian exists.

Exercise 8. (a) For a ring $A$, define $\mathbb{A}_A^n$ to be $\text{Spec } A[x_1, \ldots, x_n]$ (if $A$ is a $k$-algebra, then $\mathbb{A}_A^n$ is just $\text{Spec } A \times_{\text{Spec } k} \mathbb{A}_k^n$, but in general $\mathbb{A}_A^n$ can't be written as a product with $\mathbb{A}_k^n$ for a single field $k$). Show that $\mathbb{A}_A^n$ (together with the tuple $(x_1, \ldots, x_n)$) represents the functor $F : \text{Sch}_{\text{Spec } A} \to \text{Set}$ defined by $F(T) := \mathcal{O}_T(T)^n$ (i.e., $T$ maps to $n$-tuples of global sections of the structure sheaf of $T$).

(b) For a scheme $S$, consider the functor $F : \text{Sch}_S \to \text{Set}$ defined by $F(T) := \mathcal{O}_T(T)^n$. Show that this is representable by a scheme, which we denote $\mathbb{A}_S^n$, and call affine $n$-space.
over $S$. Hint: first show that $F$ is a Zariski sheaf, and then show that an open subscheme of $S$ gives an open subfunctor of $F$, and apply (a).

(c) Give an alternate proof of (b) by showing that the scheme $S \times_{\text{Spec} \mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$ represents $F$. 